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BOOSTER ATTITUDE STABILIZATION NETWORK SYNTHESIS

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PROPRIETARY INFORMATION

FOREWORD

This is Summary Report No. 2 on the Booster Attitude Stabilization Network Synthesis. It was prepared by Republic Aviation Corporation for the National Aeronautics and Space Administration-Marshall Space Flight Center, under NASA Contract NAS 8-5016.

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ABSTRACT 39717

This report presents the results of an investigation of a technique for synthesizing networks with resistive loads resulting in normalized element values that have voltage transfer functions meeting the gain-phase compensation requirements set by NASA. By minimization of the least squared error function using an iterative process such as the "tangent descent to a minimum" for a desk calculator, or the "gradient technique" for a digital computer, the normalized element values can be determined.

The concept of synthesizing networks from topological considerations is developed in Sections IV and V. A quadratic lag function is considered in detail in Section VI; a comparison is made between the classical synthesis technique and the one considered in this report. By considering topological configurations of interest, the mathematical process (which can be mechanized for digital computer application) yields less complex networks and non-ideal passive elements than those determined by classical synthesis techniques.

Recommendations are made for a digital computer program that should be written for 1) defining the objective function to include various network topologies of interest, and 2) mechanizing the iterative process used to determine the component values. The report suggests that additional work in specific areas may lead to simpler, more straightforward computer procedures for the realization of any network that is generally optimum in terms of its configuration and passive elements. Author

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SECTION I

INTRODUCTION

In the process of introducing stabilization networks in the attitude stabilization loop of space vehicles, design problems are often encountered that may still be considered more in the realm of the "arts" than the "sciences." In an effort to bring some form of organization to the "design art," a system approach has been considered that allows one to maintain an awareness of the ultimate circuit design objective while proceeding with the necessary calculations associated with network synthesis. By considering the problem from a "logical" point of view, it is felt that no clear demarcation between 1) approximation, and 2) realization, should exist. However, the technique suggested in this report is still a two-fold approach, with an interface relationship that allows for a simpler realization technique.

The required performance characteristics of the phase-shaping (stabilization) networks are defined in Section II. The gain or attenuation characteristics are first approximated[1] by a finite number of semi-infinite slopes, each of which in turn is closely approximated by the attenuation curve of a Butterworth or Tschebyscheff function. The resulting transfer function is then checked to determine if the phase requirements are satisfied. Depending on which is the more stringent requirement, i.e., gain or phase, it is possible to interchange the procedure outlined above. The resulting function forms a sort of interface requirement that now also has to satisfy the circuit realizability conditions. Once the interrelationships are all completely satisfied, the associated rational transfer function completely defines the allowable phase-shaping networks.

The non-unique aspect of circuit synthesis allows for a very large number of circuits, all satisfying a specified transfer function requirement. Therefore, the circuit synthesis problem may be interpreted geometrically as "the determination of a particular desirable subset from that set which satisfies the circuit requirements." The two possible approaches to solving this problem are: 1) manipulating the mathematically defined requirements until a circuit topology is derived, or 2) determining a likely circuit topology and fitting the circuit parameters to the requirements by a minimization technique.

The latter approach is used in this report, since it allows the circuit design engineer freedom to determine the network configuration and to satisfy that requirement first before proceeding to the calculation of the circuit parameters. The design engineer chooses a network topology that will satisfy the specified transfer function requirements, chooses initial values for the network element, and determines the network parameters by an iterative process that minimizes[2] the error between the specified transfer-function and that of the initially chosen network. Since only RLC elements are admitted, this approach is concerned with passive reciprocal networks without mutual inductance. A flow diagram of the design procedure is given in Figure I-1.

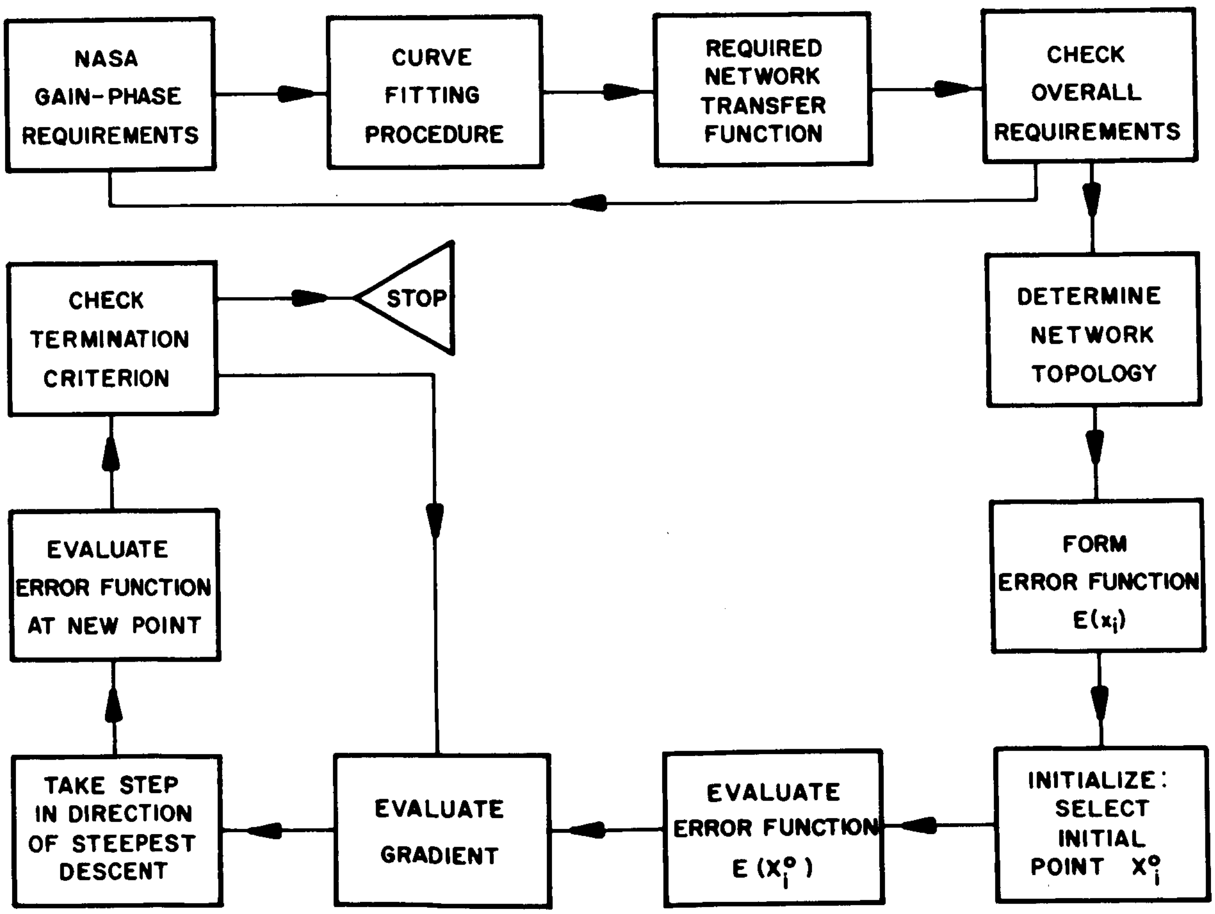


Figure I-1. Flow Diagram for Design Procedure

SECTION II

APPROXIMATION

This section considers a method of obtaining a transfer function satisfying the NASA gain-phase specifications in terms of a finite number of semi-infinite slopes on a "Bode plot," which may be considered as the straight line approximations to the continuous curve defined by Butterworth functions. The following items will be discussed in this section:

- NASA specifications
- Approximation of a NASA specification

A. NASA SPECIFICATIONS

Figures II-1 through II-12 show the NASA gain-phase specifications as bounded regions on a "Bode plot." The equation associated with each figure defines the low order solid line curve that closely satisfied the requirements. The technique used to fit the specifications may be simply described as a judicious modification of the semi-infinite slope approximations. The term "judicious modification" is used because once the form of the expression is derived, using a Butterworth or Tschebyscheff function, the break-points for the first-order terms and the damping ratio and natural frequencies of the quadratic forms must be adjusted to give a closer "fit" to the specifications.

The primary difficulty encountered in determining the equations was the fact that the various modes, as exemplified by the bounded regions, were all contained within a frequency band of about one decade width. A high-order rational function probably could be found that would satisfy both the gain and phase requirements exactly. However, it was felt that the increased complexity of the circuit would more than outweigh the small errors introduced by an inexact fit.

B. APPROXIMATION OF A NASA SPECIFICATION[3]

The routine of determining the transfer function for the particular set of specifications of Figure II-11 and Figure II-12 will now be considered, though on the surface they do not appear to be too restrictive. A semi-infinite slope approximation to the curve indicates that a minimum attenuation requirement of 24 db/octave exists between the first and second modes. Although this requirement is not too difficult to satisfy if the attenuation requirement is the only thing that has to be considered, the phase requirement imposes an added constraint that restricts the problem. As noted in Figure II-13 and Figure II-14 (log magnitude and phase diagrams for a first-order lag and a quadratic lag), the insertion effects are more pronounced on the phase angle (one decade before the corner frequency) than on the magnitude. Consequently, care must be exercised in determining the breakpoint.

From a table of factors[4] of Butterworth polynomials for n = 4 (since the semi-infinite slope approximations are in multiples of 6n db/octave), it was found that the first two resulting quadratic lag terms are:

$$\left[\left(\frac{S}{3.981}\right)^2 + 2(0.3827)\left(\frac{S}{3.981}\right) + 1\right]\left[\left(\frac{S}{3.981}\right)^2 + 2(0.9238)\left(\frac{S}{3.981}\right) + 1\right]$$

$$= Q_2\left(\frac{0.3827}{3.981}\right) Q_4\left(\frac{0.9238}{3.981}\right)$$

To turn the attenuation curve upward and at the same time introduce some leading phase angle terms to aid the phase requirements, a positive semi-infinite slope of 30 db/octave is introduced at a corner frequency of 12.59 radians. The resulting lead terms from the table of factors of Butterworth polynomials for n = 5 are:

$$\left[1 + \frac{S}{12.59}\right]\left[\left(\frac{S}{12.59}\right)^2 + 2(0.809)\left(\frac{S}{12.59}\right) + 1\right]\left[\left(\frac{S}{12.59}\right)^2\right.$$

$$\left. + 2(0.309)\left(\frac{S}{12.59}\right) + 1\right] = L_1(12.59)\; Q_1\left(\frac{0.809}{12.59}\right) Q_3\left(\frac{0.309}{12.59}\right)$$

After some adroit "adjustments," the resulting transfer function was found to be:

$$T(s) = \frac{L_1(12.59)\; Q_1\left(\frac{0.3}{12.59}\right) Q_3\left(\frac{0.8}{12.59}\right)}{Q_2\left(\frac{0.3}{3.981}\right) Q_4\left(\frac{Q.4}{3.981}\right) Q_6\left(\frac{0.7}{500}\right)} \qquad \text{(II-1)}$$

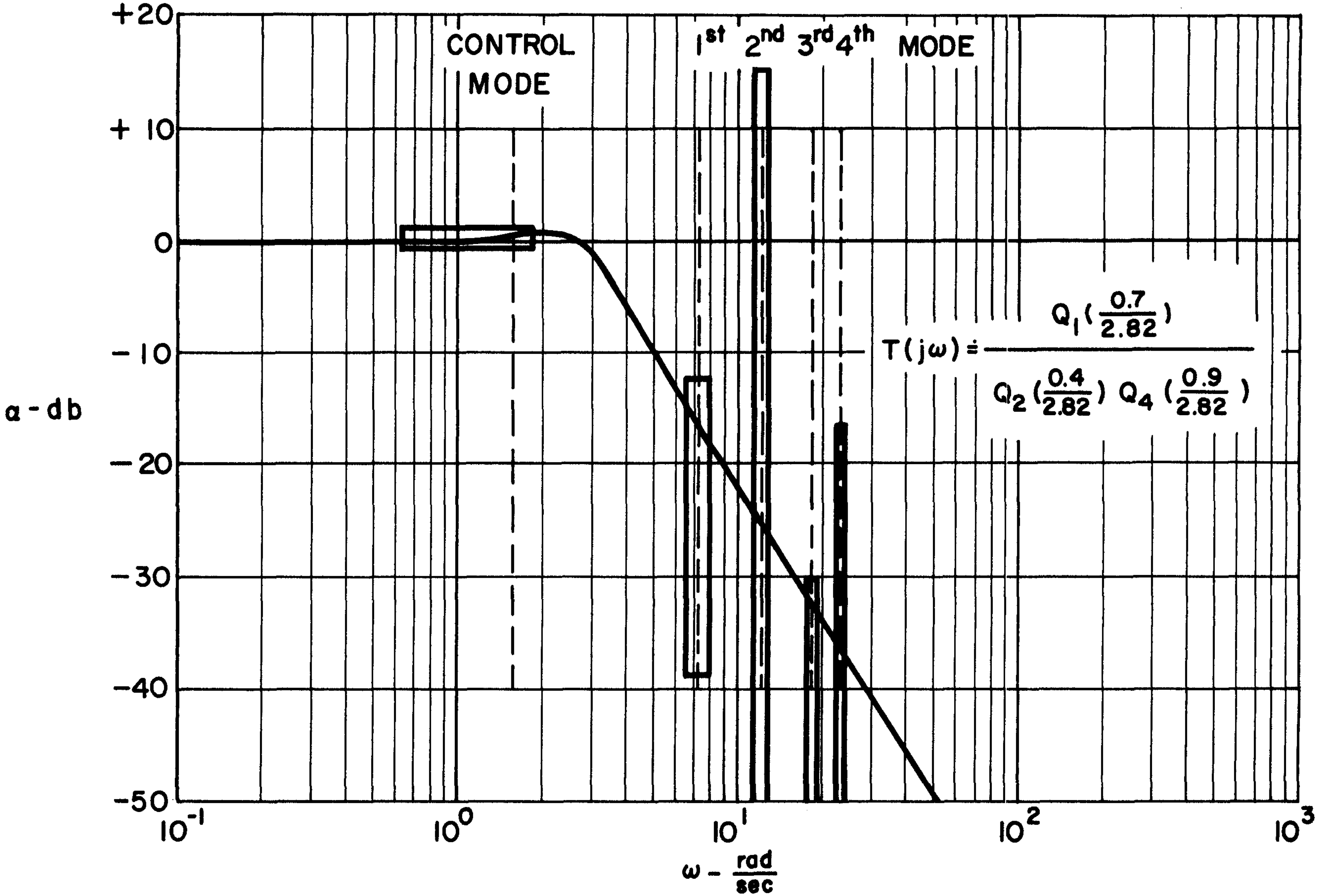


Figure II-1. Gain Tolerance for Rate Gyro Filter (fit for gain)

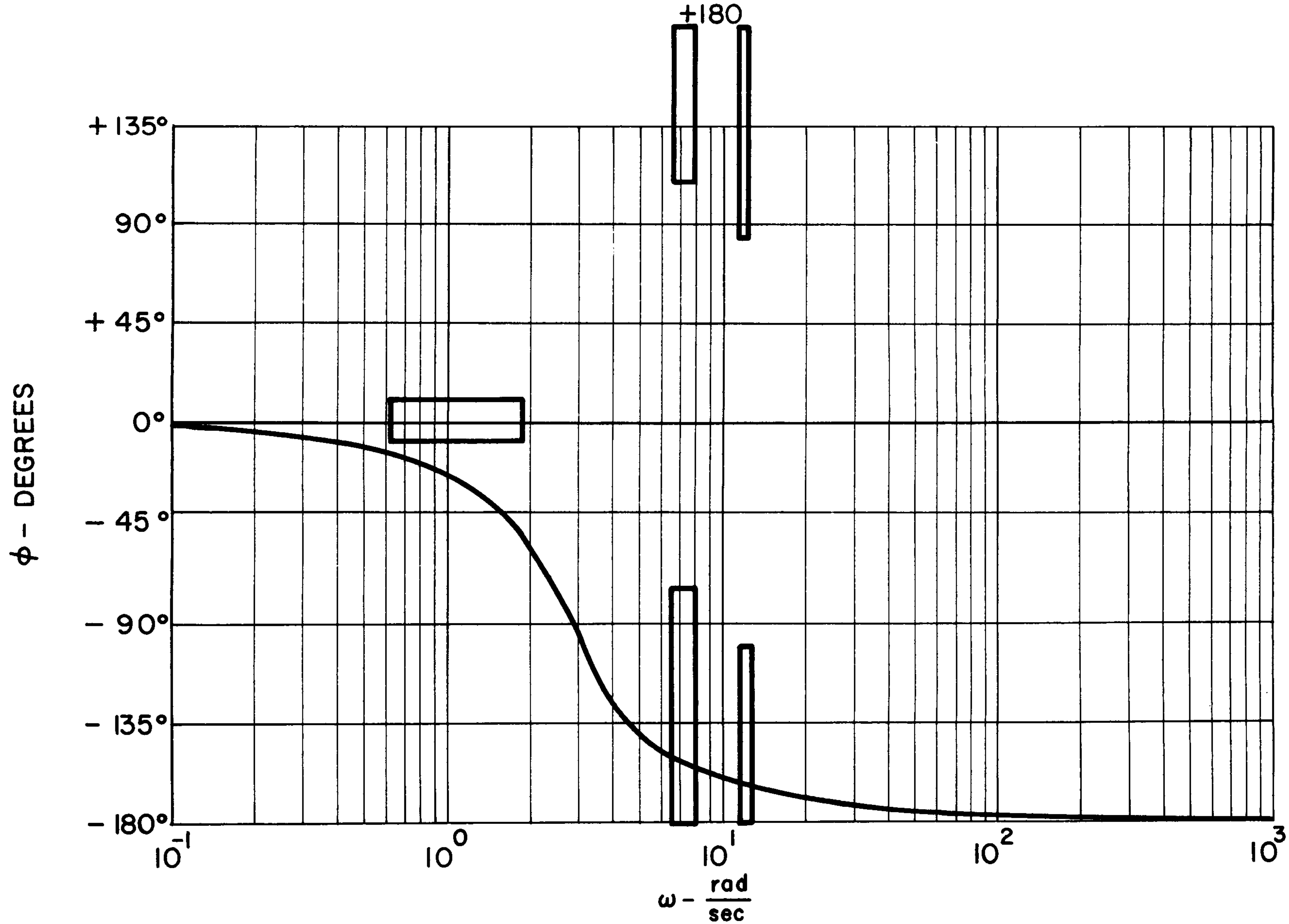


Figure II-2. Phase Tolerance for Rate Gyro Filter (fit for gain)

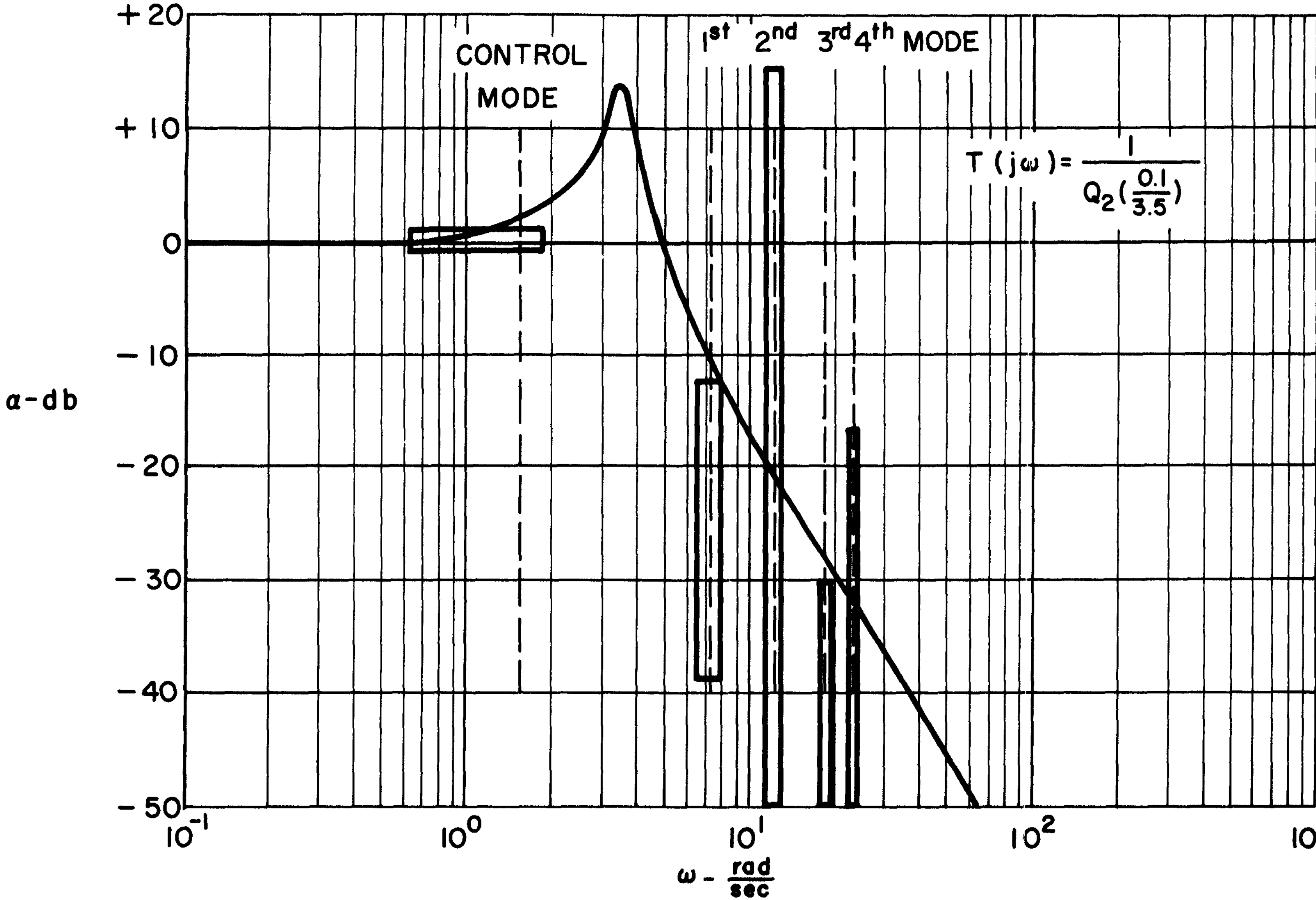
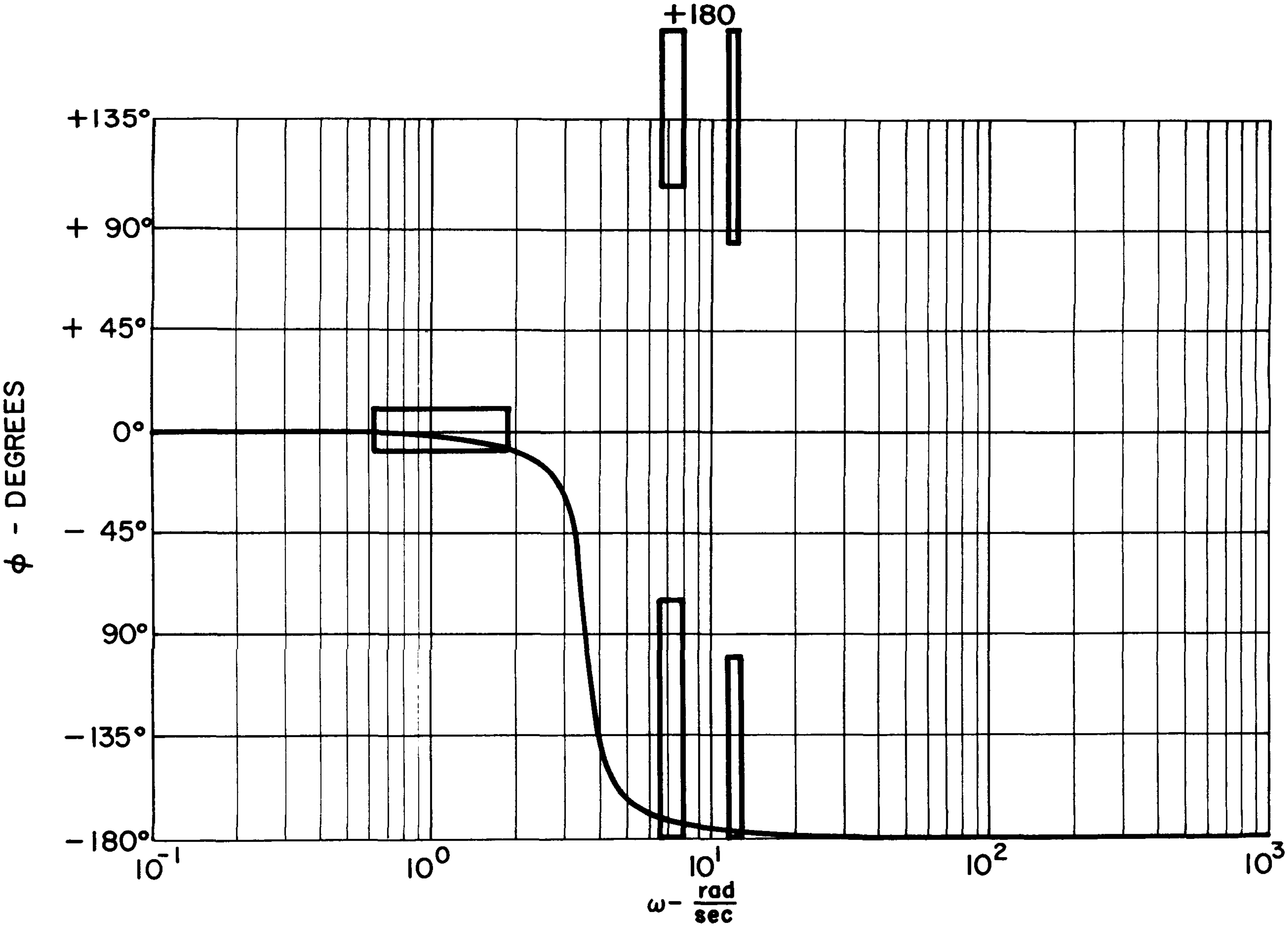


Figure II-3. Gain Tolerance for Rate Gyro Filter (fit for phase)



... for Rate Gyro Filter (fit for phase)

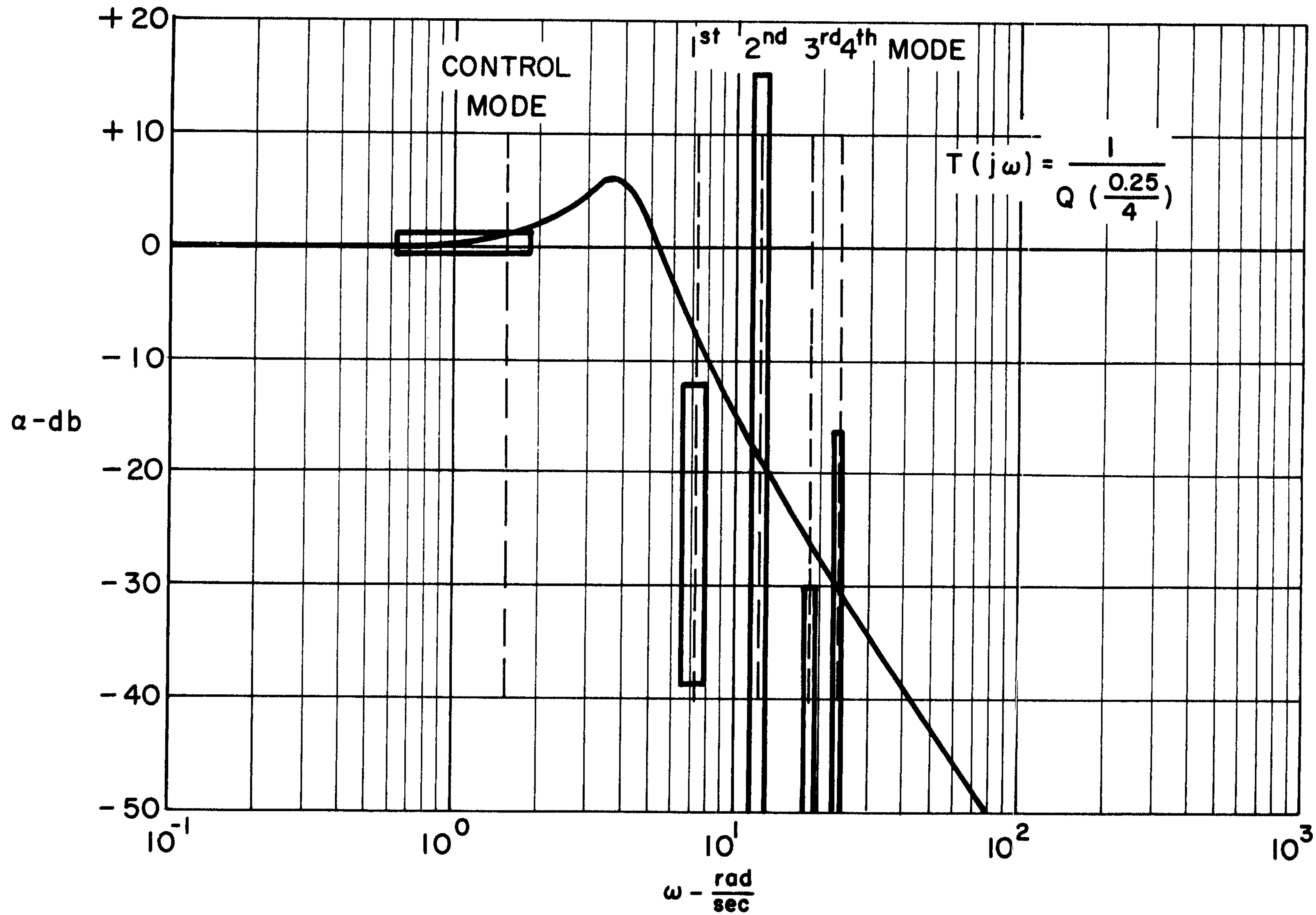


Figure II-5. Gain Tolerance for Rate Gyro Filter (compromise fit - gain and phase)

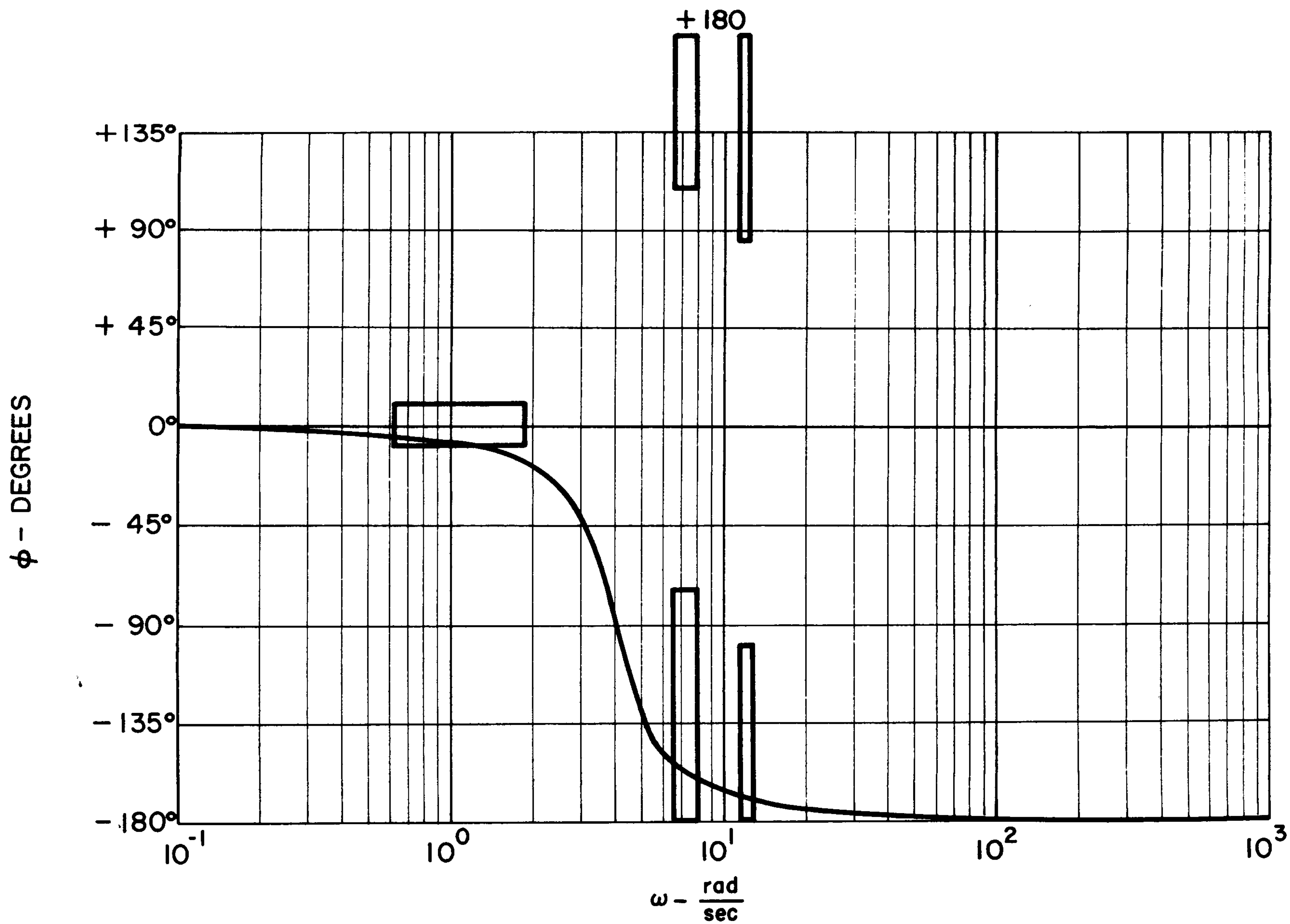


Figure II-6. Phase Tolerance for Rate Gyro Filter (compromise fit - gain and phase)

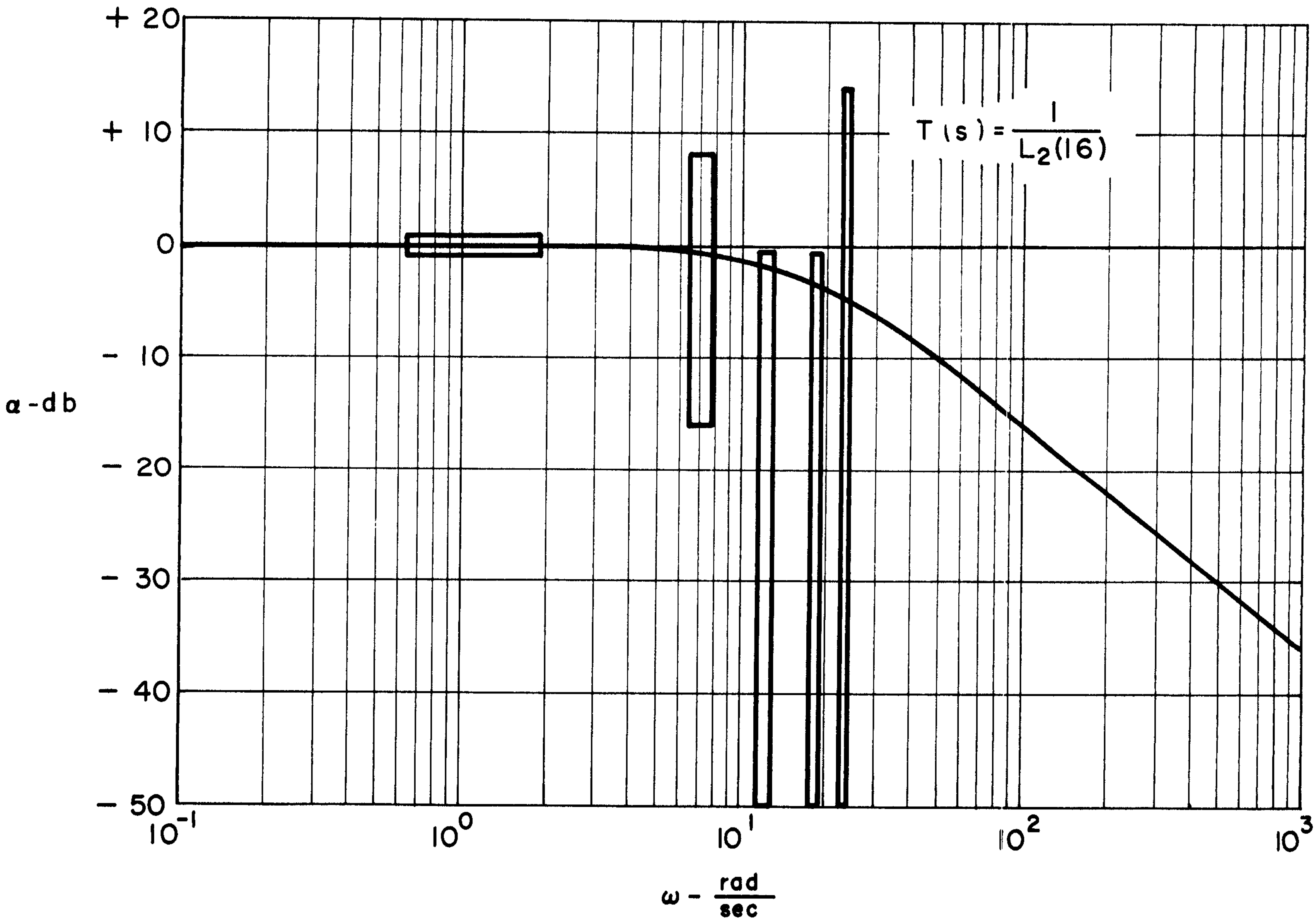


Figure II-7. Gain Tolerance for Attitude Gyro Filter (fit for gain and phase)

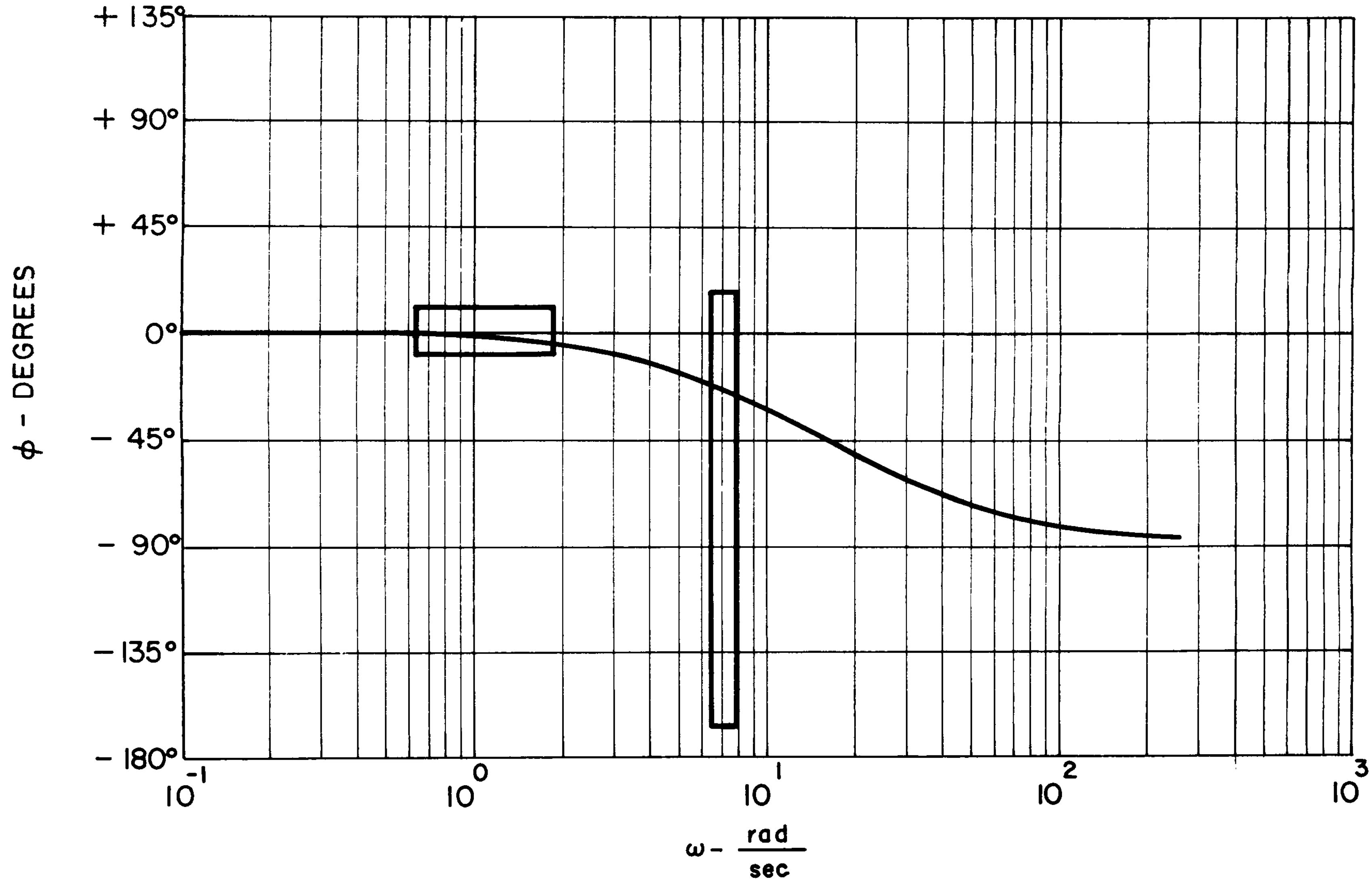


Figure II-8. Phase Tolerance for Attitude Gyro Filter (fit for gain and phase)

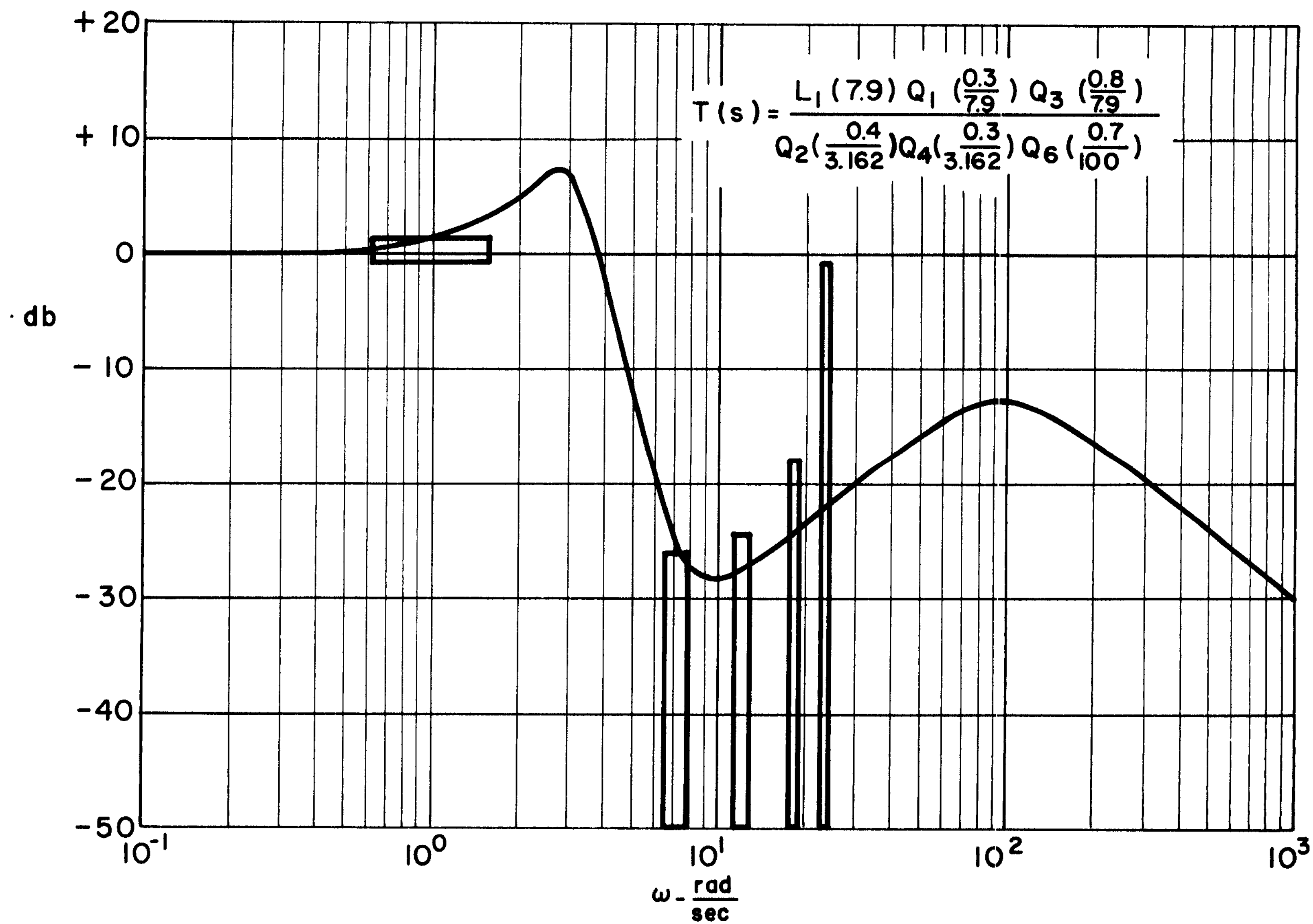


Figure II-9. Gain Tolerance for α- Meter Filter (compromise fit - gain and phase)

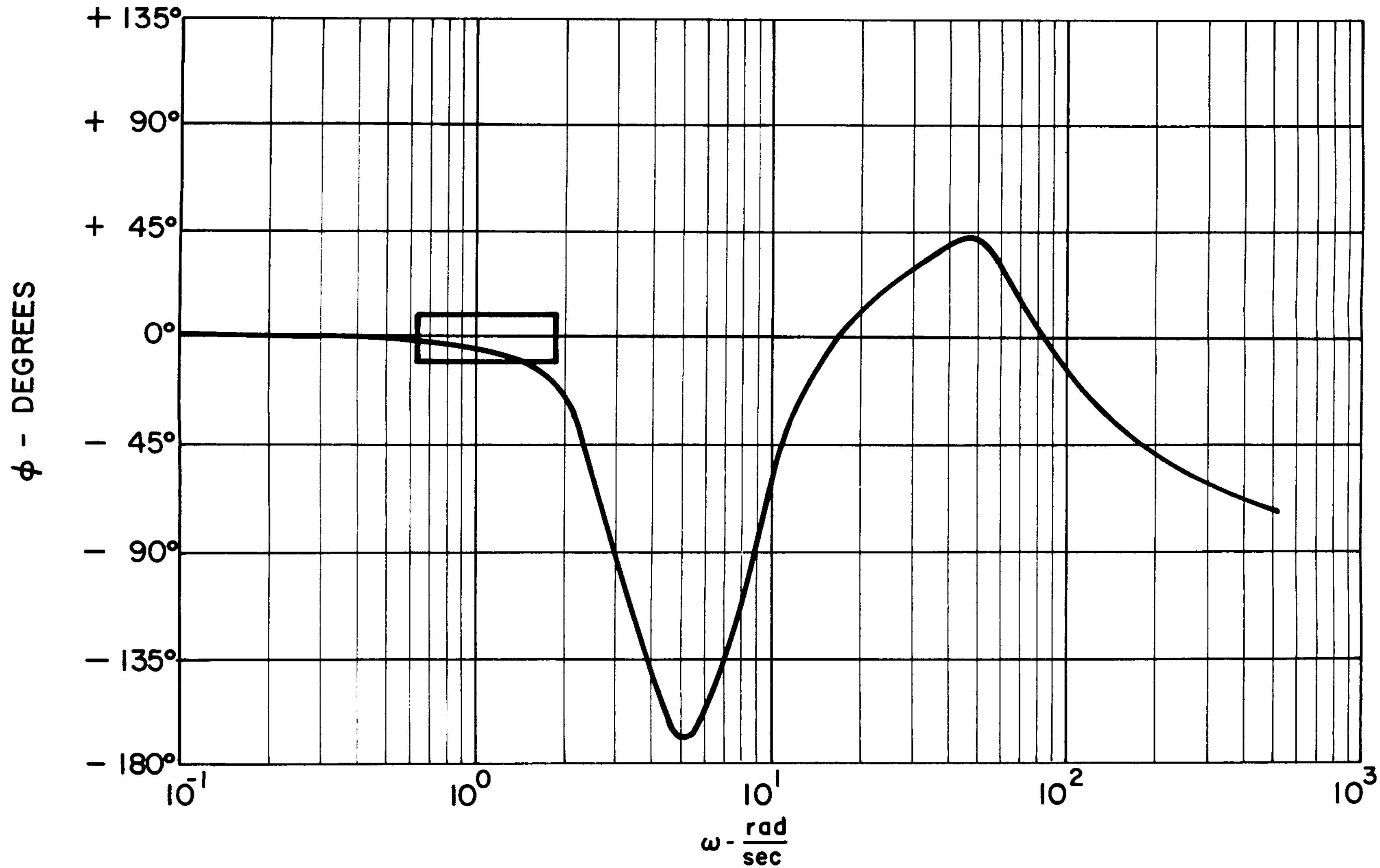


Figure II-10. Phase Tolerance for α- Meter Filter (compromise fit - gain and phase)

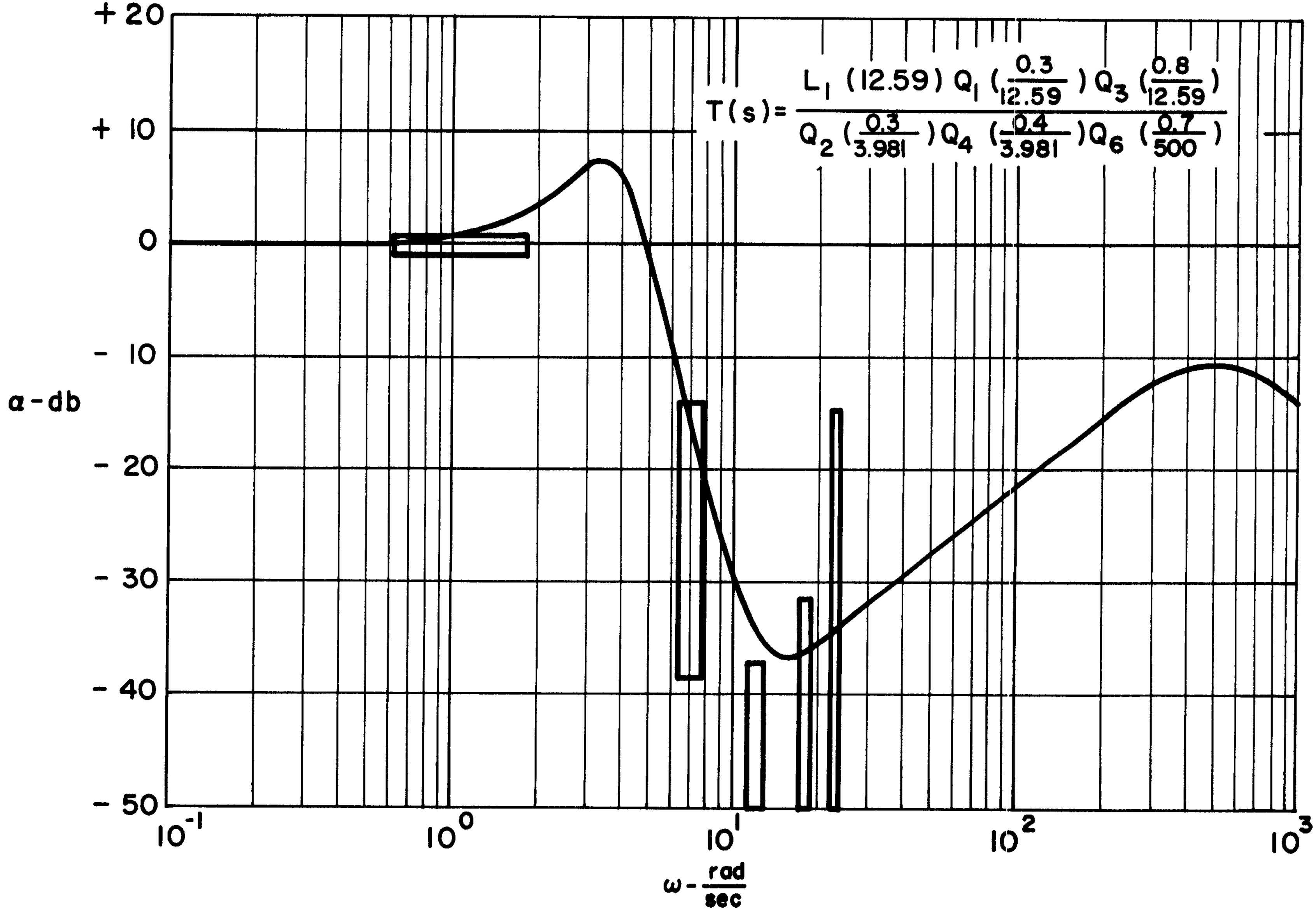


Figure II-11. Gain Tolerance for Accelerometer Filter (compromise fit - gain and phase)

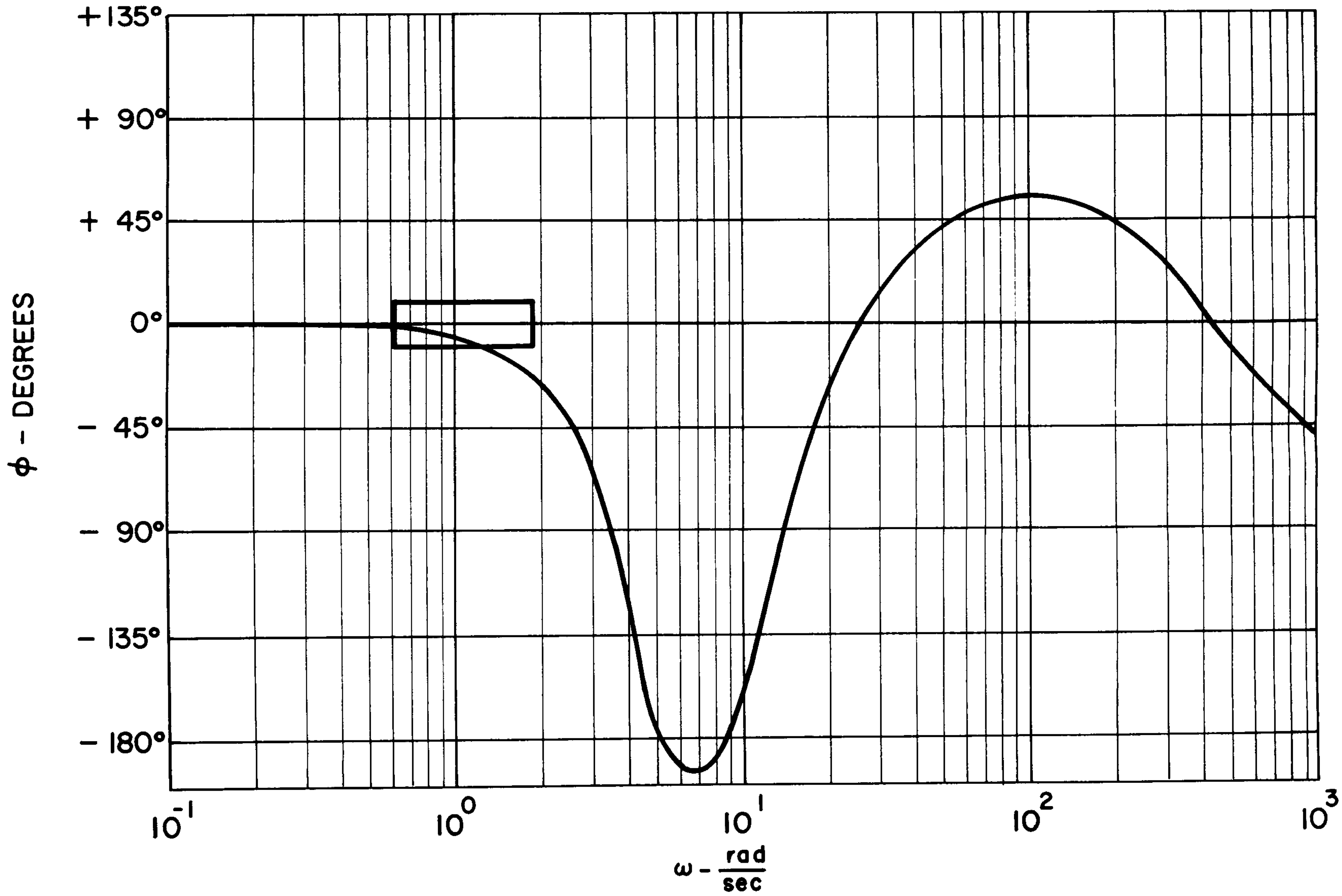


Figure II-12. Phase Tolerance for Accelerometer Filter (compromise fit - gain and phase) with the quadratic lag, $Q_6\left(\frac{0.7}{500}\right)$ being introduced outside the system response specification to satisfy the realizability requirements.

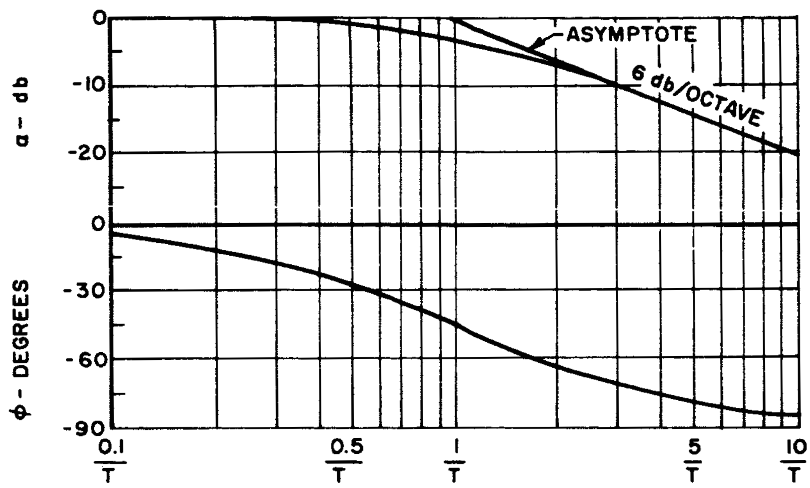


Figure II-13. Log Magnitude and Phase Plots for a Simple First Order Lag Function

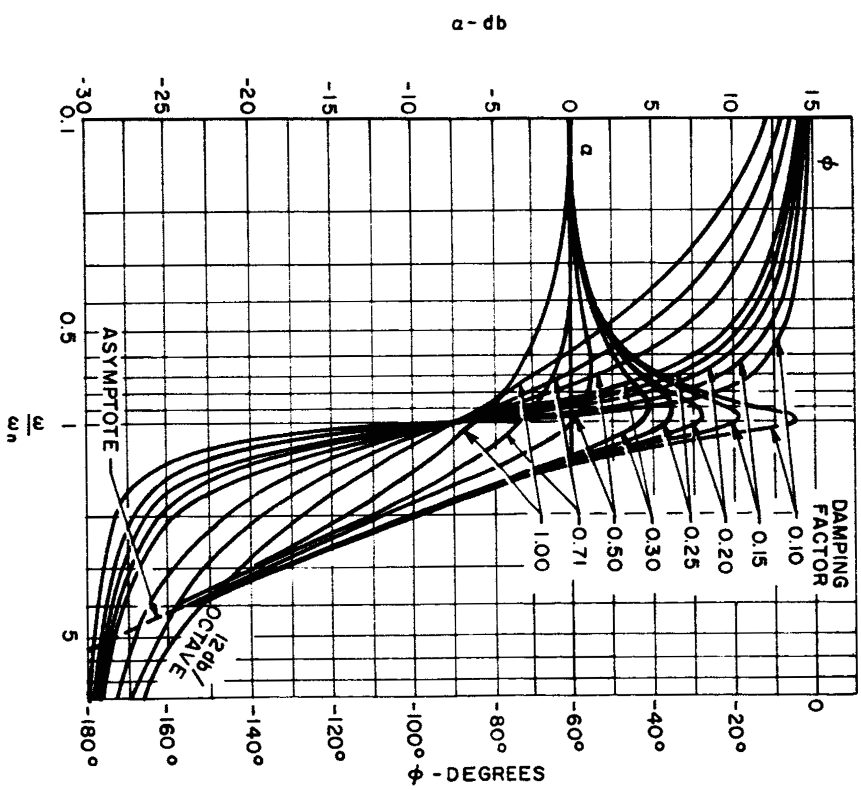


Figure II-14. Log Magnitude and Phase Plot for Quadratic Lag Functions with Various Damping Coefficients

SECTION III

DESIGN CRITERION

The method of synthesis from the topological point of view[5,6] requires the adoption of a criterion to determine when a particular network transfer function t(s) "best" approximates the given transfer function T(s). An error

$$\epsilon(s) = T(s) - t(s) \qquad \text{(III-1)}$$

will exist whenever the two transfer functions are not identical.

The simple expedient of considering the square of the error would produce a meaningful measure, since it would eliminate the problem of negative and positive errors negating each other. Specifically, the two functions developed in this section

$$E = \sum_{s=o}^{t} \left[R(\omega_s)\Delta_R(\omega_s) - I(\omega_s)\Delta_I(\omega_s) - n_R(\omega_s)\right]^2 + \left[R(\omega_s)\Delta_I(\omega_s) + I(\omega_s)\Delta_R(\omega_s) - n_I(\omega_s)\right]^2 \qquad \text{(III-2)}$$

and

$$\Phi = \sum_{s=1}^{t} \left[R(\omega_s) - R_o(\omega_s)\right]^2 + \left[I(\omega_s) - I_o(\omega_s)\right]^2 \qquad \text{(III-3)}$$

consider a sum-of-squares criterion as the useful measure of the error between the two transfer functions. Each of the above error functions states that over the whole spectrum,[7] at those discrete points of interest, some error (as indicated by the above relations) will exist.

As with all error criteria or grading systems, it must be considered on an individual basis and satisfy the particular requirement of the situation. Both error Equations (III-2 and III-3) are developed and presented, although the first one was used extensively for the calculation of the network parameters. Since both are similar and valid definitions of a form of the sum of square error criterion, for the purpose of clarity, a distinction is made between them by calling the first error function E and the second error function Φ. The following items will be discussed in this section:

- E Error criterion
- Φ Error criterion

A. ERROR CRITERION

From the curve approximation, the transfer function is defined in general form as:

$$T(s) = \frac{\sum_{i=0}^{n} N_i S^i}{\sum_{j=0}^{m} D_j S^j} = \frac{N_o + N_1 S + N_2 S^2 + ---- + N_n S^n}{D_o + D_1 S + D_2 S^2 + --- + D_m S^m} \tag{III-4}$$

At the discrete frequencies of interest, the above may be evaluated in terms of its real and imaginary components as:

$$T(j\omega_s) = R(\omega_s) + j\, I(\omega_s) \tag{III-5}$$

The transfer function derived from a particular topology can be defined in the frequency domain as:

$$t(j\omega_s) = \frac{n_R(\omega_s) + j\, n_I(\omega_s)}{\Delta_R(\omega_s) + j\, \Delta_I(\omega_s)} \tag{III-6}$$

where:

$n_R(\omega_s)$ = real component of numerator

$n_I(\omega_s)$ = imaginary component of numerator

$\Delta_R(\omega_s)$ = real component of denominator

$\Delta_I(\omega_s)$ = imaginary component of denominator

Equations III-12 and III-13 are used in the linearized approximations to determine the increments added to each variable x_i at each point in the iterative process.

B. ERROR FUNCTION

From the given NASA graphical performance specification, the minimum-phase physically realizable transfer function can be represented by the rational function of the complex frequency variable as

$$T(s) = \frac{E_{OUT}}{E_{IN}}(s) = \frac{A_o + A_1S + A_2S^2 + \ldots\ldots\ldots A_nS^n}{B_o + B_1S + B_2S^2 + \ldots\ldots\ldots B_mS^m} = \frac{\sum_{i=0}^{n} A_iS^i}{\sum_{j=0}^{m} B_jS^j} \qquad \text{(III-14)}$$

Separating the two rational polynomials of Equation (III-14) in terms of M and N (their even and odd parts, respectively) results in

$$T(s) = \frac{M_1+N_1}{M_2+N_2} = \frac{(A_o+A_2S^2+\ldots.)+(A_1S+A_3S^3+\ldots.)}{(B_o+B_2S^2+\ldots.)+(B_1S+B_3S^3+\ldots.)} \qquad \text{(III-15)}$$

Rationalization of Equation III-15 results in

$$T(s) = \frac{M_1+N_1}{M_2+N_2} \times \frac{M_2-N_2}{M_2-N_2} = \frac{M_1M_2-N_1N_2}{M_2^2-N_2^2} + \frac{N_1M_2-N_2M_1}{M_2^2-N_2^2} \qquad \text{(III-16)}$$

where the first term on the right is an even function of s and the second term on the right is an odd function of s.

Thus, for a set of discrete angular real frequencies, ω_r, $(r = 1, 2, \ldots m)$, Equation (III-16) can be written in terms of real and imaginary components suitable in the manipulations of subsequent work, as:

$$T(j\omega_r) = \left.\frac{M_1M_2 - N_1N_2}{M_2^2 - N_2^2}\right|_{s=j\omega_r} + \left.\frac{N_1M_2 - N_2M_1}{M_2^2 - N_2^2}\right|_{s=j\omega_r} = R(\omega_r) + jI(\omega_r) \qquad \text{(III-17)}$$

The ensuing mathematical development will describe a numerical technique for synthesizing a network with imposed constraints for a specific application.

The requirements for the control application under consideration are to synthesize a linear, passive, reciprocal, transformerless three-terminal network (3T.N.) with given terminations, the transfer function of which closely approximates Equation (III - 14) in its coefficient values.

The approximating transfer function obtained from that class of networks containing the desired topology is:

$$T_o(s) = \frac{a_o + a_1 s + a_2 s^2 + \ldots\ldots . a_n s^n}{b_o + b_1 s + b_2 s^2 + \ldots\ldots . b_m s^m} = \frac{\sum_{i=0}^{n} a_i s^i}{\sum_{j=0}^{m} b_j s^j} \qquad \text{(III-18)}$$

where the summation running variable is of the same order as that of Equation (III-14). Equation (III-18) will, in general, be nonlinear in both the discrete real frequency variable (which may or may not be periodic) and its coefficients, the available dependent variable parameters, which are normally multilinear functions of the element values of the network. For convenience of notation, the dependent variables are defined as x_i where $i = 1, 2, \ldots . n$. Rewriting Equation (III-18) in the form to coincide with Equation (III-17), we obtain,

$$T_o(j\omega_r) = \left.\frac{m_1m_2 - n_1n_2}{m_2^2 - n_2^2}\right|_{s=j\omega_r} + \left.\frac{n_1m_2 - n_2m_1}{m_2^2 - n_2^2}\right|_{s=j\omega_r} = R_o(\omega_r) + jI_o(\omega_r) \qquad \text{(III-19)}$$

For $r = 1, 2, \ldots . n$

Network synthesis being non-unique, there are an infinite number of circuits satisfying Equation (III-14), but since we have imposed topological constraints, our investigation is limited to a finite number. Even so, the element values in a single circuit satisfying the constraints may take on any number of possible combinations of values and still satisfy the terminal conditions. The objective is to determine any set of element values (R_i, L_i, C_i) contained in Equation (III-18) which approximates Equation (III-14), the set to consist of realistic values. The unknown element values of Equation (III-18) approximating Equation (III-14) are determined by the minimization of the sum-of-squares error, i.e., optimization in the least squares sense. This error criterion was chosen on the basis of being mathematically the most tractable problem.

The principle of least squares states that the most desirable values of the unknown parameters are those for which the sum of the squares of the errors (differences between approximating and desired functions) is a minimum.

The sum-of-squares error is derived as follows:

Form the numerical difference between Equations (III-17) and (III-19):

$$\epsilon(\omega_r) = T(j\omega_r) - T_o(j\omega_r) = \left[R(\omega_r) - R_o(\omega_r)\right] + j\left[I(\omega_r) - I_o(\omega_r)\right] \qquad \text{(III-20)}$$

Take the absolute value of both sides:

$$\begin{aligned} |\epsilon(\omega_r)| &= \left|\left[R(\omega_r) - R_o(\omega_r)\right] + j\left[I(\omega_r) - I_o(\omega_r)\right]\right| \\ &= \left\{\left[R(\omega_r) - R_o(\omega_r)\right]^2 + \left[I(\omega_r) - I_o(\omega_r)\right]^2\right\}^{\frac{1}{2}} \end{aligned} \qquad \text{(III-21)}$$

Squaring both sides of Equation (III-21) results in:

$$|\epsilon(\omega_r)|^2 = \left[R(\omega_r) - R_o(\omega_r)\right]^2 + \left[I(\omega_r) - I_o(\omega_r)\right]^2 \qquad \text{(III-22)}$$

It is this quantity in Equation (III-22) redefined as

$$\Phi = |\epsilon(\omega_r)|^2 = \sum_{r=1}^{m} \left\{\left[R(\omega_r) - R_o(\omega_r)\right]^2 + \left[I(\omega_r) - I_o(\omega_r)\right]^2\right. \qquad \text{(III-23)}$$

which is to be minimized.

SECTION IV

CIRCUIT TOPOLOGY

In this section, a description of the networks is considered in terms of their physical geometry or configuration; it consequently has the generic title of network "topology" as the all-inclusive property. Since only the basic concept of the mathematics of topology is necessary to yield interesting and useful results in the application of network synthesis, the following items will be discussed in this section:

- Graph Concept
- Circuit Equations

A. GRAPH CONCEPT[9]

When considering an electrical network from a geometric or topological point of view, its graph is of great importance. A graph is a diagram representing the structural framework of the network; it is found by replacing each of the circuit elements by line segments, with each line segment connecting two vertices, or nodes.

A branch is a line segment of a graph, including its two vertices. Its length or curvature has no meaning; only the vertices it connects are important. Each network element, i.e., resistance, capacitance, or inductance, could be considered as a branch; or a complicated combination of network elements could be considered as a branch. Mutual inductances are excluded from this consideration and consequently are not defined. In drawing a graph all sources are removed, with the voltages short-circuited and the current open-circuited. Figure IV-1(a) is an example of an electrical network and its associated graph, Figure IV-1(b).

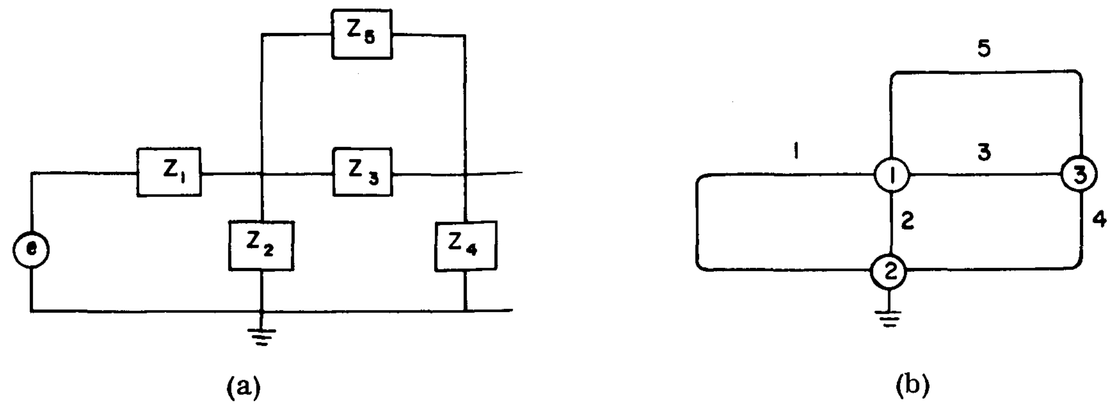


Figure IV-1. Electrical Network and Its Associated Graph

In summary, in the graph concept the topology of the network is important, i.e., what branch connects which node pair. A mathematical description is possible by formulating an array having one column for each branch and one row for each node. In each row of the array a +1 is placed in the column corresponding to a branch termination, a -1 is placed in the column corresponding to a branch origin, and zero if neither of the preceding two conditions is satisfied. The array so generated is termed the node-incidence matrix or vertex-incidence matrix of the graph (I_n). Another topological description of the graph is possible, (leading to a dual development) by noting the incidence of the branches on the loops, then formulating an array having one column for each branch and one row for each loop (or mesh). In each row of the array, a +1 is placed in the column corresponding to a branch direction the same as that traversed by the loop; a -1 is placed in the column corresponding to a branch direction opposite to the one traversed by the loop; and zero is used if the branch is not contained in the loop. The array so generated is termed the branch-mesh incidence matrix of the graph (I_b). Since the latter description is the only one used in the present work, it will be simply designated by the capital letter "I" without any subscript; all subsequent derivations will be directed along this line.

The graph concept allows one to formulate Kirchhoff's basic circuit laws in concise matrix notation as:[10]

1) Kirchhoff's current law (KCL) - The summation of all the branch circuits at a node, by the continuity condition, must be zero.

$$\sum_p i_p - \sum_q i_q = 0$$

p represents all branches terminating at node n

q represents all branches originating at node n

In matrix notation this can be written as:

$$[I_n]\ [i\ (t)] = 0 \qquad \text{(IV-1)}$$

where $[I_n]$ = node-incidence matrix

$[i\ (t)]$ = column matrix of branch currents.

2) Kirchhoff's voltage law (KVL) - The summation of the branch voltages around a closed loop, must be zero.

$$\sum_p v_p - \sum_q v_q = 0 \qquad \text{(IV-2)}$$

p represents all branch voltages in direction of loop traverse

q represents all branch voltages in the opposite direction of loop traverse

In matrix notation this can be written as:

$$[I]\ [v\ (t)] = 0 \qquad \text{(IV-3)}$$

where $[I]$ = branch mesh incidence matrix

$[v\ (t)]$ = column matrix of branch voltages.

KVL in terms of the branch voltages is given by:

$$[I]\ [v\ (t)] = [e_e(t)] \qquad \text{(IV-4)}$$

where $[e_e(t)]$ is the column matrix, each of whose elements gives the total source voltage rise in the corresponding loop.

Now: $$[i_{br}(t)] = [I]'\ [i_e(t)] \qquad \text{(IV-5)}$$

but $$[v_{br}(t)] = [Z_{br}]\ [i_{br}(t)] \qquad \text{(IV-6)}$$

$$= [Z_{br}]\ [I]'\ [i_e(t)] \qquad \text{(IV-7)}$$

$$\therefore\ [I]\ [Z_{br}]\ [I]'\ [i_e(t)] = [e_e(t)] \qquad \text{(IV-8)}$$

or $[Z][i_e(t)] = [e_e(t)]$ (IV-9)

where $[Z] = [I][Z_{br}][I]'$

Where Z in Equation (IV-9) represents the impedance matrix for a particular loop.

Adapting the Kron[11] convention of referring to the diagonal matrix $[Z_{br}]$ as the primitive network impedance matrix or simply the "primitive impedance matrix," it is possible to determine the impedance matrix for the network once the incidence matrix for the topology is known.

As an example of the usefulness of the "primitive matrix" concept, consider the development of the network impedance $[Z]$ for the three-loop five-branch network shown in Table IV-2, Figure 5. From Equation IV-9

$$[Z] = [I][Z_{br}][I]'$$

therefore the characteristic impedance[12] may be determined by matrix multiplication by considering the expression

$$[Z] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z_1 & 0 & 0 & 0 & 0 \\ 0 & Z_2 & 0 & 0 & 0 \\ 0 & 0 & Z_3 & 0 & 0 \\ 0 & 0 & 0 & Z_4 & 0 \\ 0 & 0 & 0 & 0 & Z_5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

resulting in:

$$[Z] = \begin{bmatrix} (Z_1+Z_2) & -Z_2 & 0 \\ -Z_2 & (Z_2+Z_3+Z_4) & -Z_3 \\ 0 & -Z_3 & (Z_3+Z_5) \end{bmatrix}$$

B. CIRCUIT EQUATIONS

Considering a generalized n-mesh linear network, it is possible to write the "equations of motion" describing its behavior as:[13]

$$E_1 = Z_{11}i_1 + Z_{12}i_2 + Z_{13}i_3 + \text{-----------} + Z_{1n}i_n \quad \text{(IV-10)}$$

$$E_2 = Z_{21}i_1 + Z_{22}i_2 + Z_{23}i_3 + \text{-----------} + Z_{2n}i_n$$

$$E_3 = Z_{31}\, i_3 + Z_{32} i_2 + Z_{33} i_3 + \text{-----------} + Z_{3n} i_n$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$E_n = Z_{n1} i_1 + Z_{n2} i_2 + Z_{n3} i_3 + \text{-----------} + Z_{nn}\, i_n$$

E_1, E_2, . . . E_n represent the voltage, or the forcing or disturbing function applied to the contours of the mesh under consideration. By the principle of superposition, the total effect of the voltages applied to the network is equal to the sum of the effects of each individual voltage applied individually. For this report, without losing any generality, only one forcing function is considered, with all the others being equal to zero: $E_1 = E$. Therefore the above equations in matrix form may be written as:

$$\begin{bmatrix} E_1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & \text{--------} & Z_{1n} \\ Z_{21} & Z_{22} & Z_{23} & \text{--------} & Z_{2n} \\ Z_{31} & Z_{32} & Z_{33} & \text{--------} & Z_{3n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ Z_{n1} & Z_{n2} & Z_{n3} & & Z_{nn} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ \cdot \\ \cdot \\ \cdot \\ i_n \end{bmatrix} \qquad \text{(IV-11)}$$

or:

$$[E] = [Z]\,[i] \qquad \text{(IV-12)}$$

The elements Z_{ij}, are in general complex quantities, where:

$$Z_{ij} = \begin{cases} i = j & \text{Represents mesh self-impedances} \\ i \neq j & \text{Represents mesh mutual impedances} \end{cases} \qquad \text{(IV-13)}$$

Therefore the current through the k^{th} loop may be determined as:

$$i_k = \frac{|M|E_1}{|Z|} \quad \text{and} \qquad \text{(IV-14)}$$

the voltage drop across an impedance in the k^{th} loop is:

$$e_k = Z_k\, i_k \qquad \text{(IV-15)}$$

resulting in a transfer function:

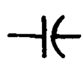
$$\frac{e_k}{E_1} = t\,(s) = \frac{|M|\; Z_k}{|Z|} \qquad \text{(IV-16)}$$

$|M|$ represents the minor of the determinant $|Z|$, resulting from the cancelling of that column which belongs to the variable of interest, and of that row which corresponds in Equation (IV-11) to the expression with the non-zero right side.

The determinant of the impedance matrix Z developed in Equation (IV-9) is identical to the one in Equation (IV-14), therefore there is a definite link between the topological development of the impedance function and the circuit development.

Table IV-1 summarizes the five admissible[14] branches considered in this work, as making up the primitive impedance matrix [Z].

TABLE IV-1

ADMISSIBLE BRANCHES	
Branch Configuration	Impedance Function (Z_i)
	$R + LS + \frac{1}{CS}$
	$R + LS$
	$R + \frac{1}{CS}$
	R
	$\frac{1}{CS}$

It may be noted that only practical branch configurations are considered, i.e., configurations with dissipative elements. The one contradiction to this statement is allowing for the existence of an ideal capacitor as a branch element; however, it was felt that for all practical purposes the capacitor does exist as an ideal element. Figure IV-2 is the generalized network-graph and incidence array from which all the two and three mesh admissible topologies considered were generated as summarized in Table IV-2. The specific networks considered are represented in Figures IV-3 through IV-17, with the general network transfer function.

TABLE IV-2. ADMISSIBLE ONE, TWO, AND THREE MESH NETWORKS

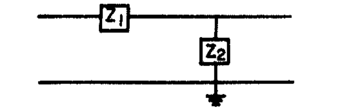
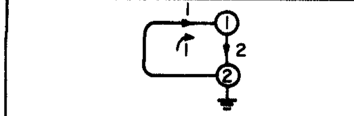
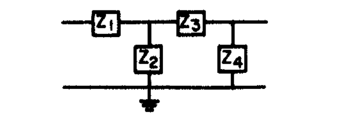
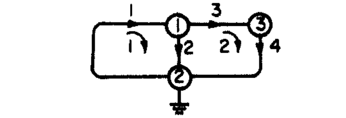
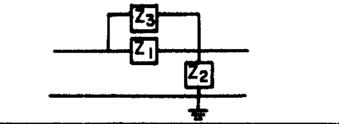
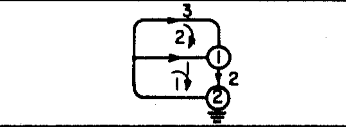
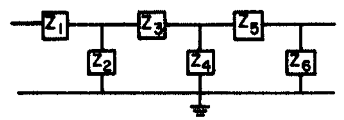
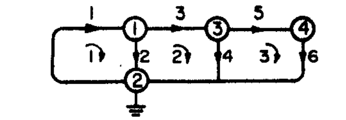
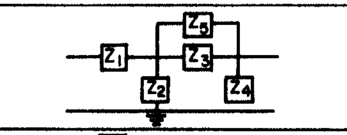
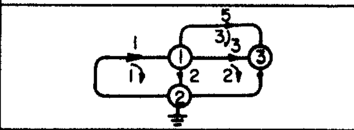
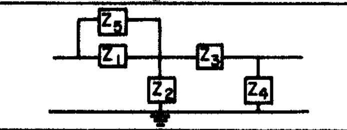
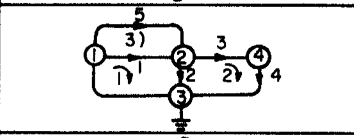
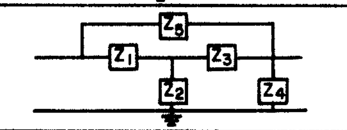
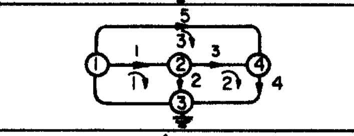
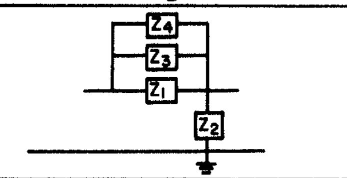
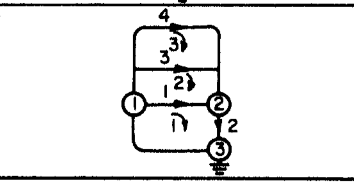
	ADMISSIBLE NETWORK	TOPOLOGICAL GRAPH	INCIDENCE MATRIX (meshes \ branches)	NETWORK IMPEDANCE $[Z]$
1			$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \end{array}$	$\begin{bmatrix} Z_1 + Z_2 \end{bmatrix}$
2			$\begin{array}{c\|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 1 \end{array}$	$\begin{bmatrix} (Z_1+Z_2) & -Z_2 \\ -Z_2 & (Z_2+Z_3+Z_4) \end{bmatrix}$
3			$\begin{array}{c\|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}$	$\begin{bmatrix} (Z_1+Z_2) & -Z_1 \\ -Z_1 & (Z_1+Z_3) \end{bmatrix}$
4			$\begin{array}{c\|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & -1 & 1 & 1 \end{array}$	$\begin{bmatrix} (Z_1+Z_2) & -Z_2 & 0 \\ -Z_2 & (Z_2+Z_3+Z_4) & -Z_4 \\ 0 & -Z_4 & (Z_4+Z_5+Z_6) \end{bmatrix}$
5			$\begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 1 & 1 & 0 \\ 3 & 0 & 0 & -1 & 0 & 1 \end{array}$	$\begin{bmatrix} (Z_1+Z_2) & -Z_2 & 0 \\ -Z_2 & (Z_2+Z_3+Z_4) & -Z_3 \\ 0 & -Z_3 & (Z_3+Z_5) \end{bmatrix}$
6			$\begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 0 & 0 & 1 \end{array}$	$\begin{bmatrix} (Z_1+Z_2) & -Z_2 & -Z_1 \\ -Z_2 & (Z_2+Z_3+Z_4) & 0 \\ -Z_1 & 0 & (Z_1+Z_5) \end{bmatrix}$
7			$\begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & -1 & 0 & 1 \end{array}$	$\begin{bmatrix} (Z_1+Z_2) & -Z_2 & -Z_1 \\ -Z_2 & (Z_2+Z_3+Z_4) & -Z_3 \\ -Z_1 & -Z_3 & (Z_1+Z_3+Z_5) \end{bmatrix}$
8			$\begin{array}{c\|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & +1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & -1 & 1 \end{array}$	$\begin{bmatrix} (Z_1+Z_2) & -Z_1 & 0 \\ -Z_1 & (Z_1+Z_3) & -Z_3 \\ 0 & -Z_3 & (Z_3+Z_4) \end{bmatrix}$

TABLE IV-3. SUMMARY OF DERIVED TRANSFER FUNCTION

$$T(s) = \frac{\sum_{n=o}^{i} N_n S^n}{\sum_{m=o}^{j} D_m S^m}$$

i	j	Figure	Title
0	1	IV-4	Resistive Load, One Loop, 3-Element Circuit
2	2	IV-3	One Loop, 5-Element Circuit
2	3	IV-6	Resistive Load, Two-Loop, 8-Element Circuit
2	3	IV-8	Resistive Load, Two-Loop, 6-Element Circuit
4	4	IV-5	Two Loop, 10-Element Circuit
4	4	IV-7	Two Loop, 8-Element Circuit
4	5	IV-10	Resistive Load, Three Loop, 13-Element Circuit
4	5	IV-12	Resistive Load, Three Loop, 11-Element Circuit
4	5	IV-14	Resistive Load, Three Loop, 11-Element Circuit
4	5	IV-15aa	Resistive Load, Three Loop, 12-Element Circuit
4	5	IV-15bb	Resistive Load, Three Loop, 11-Element Circuit
4	5	IV-17	Resistive Load, Three Loop, 9-Element Circuit
6	6	IV-9	Three Loop, 15-Element Circuit
6	6	IV-11	Three Loop, 13-Element Circuit
6	6	IV-13	Three Loop, 13-Element Circuit
6	6	IV-15a	Three Loop, 14-Element Circuit
6	6	IV-15b	Three Loop, 13-Element Circuit
6	6	IV-16	Three Loop, 11-Element Circuit

NETWORKS										BRANCHES 1	2	3	4	5	6	7	8	9	10
										1	1								
										2	2	2	2						
										3	3					3			
										4	4	4	4	4	4				
										5	5	5	5				5	5	
										6	6	6	6			6	6		
										7	7	7	7			7	7	7	
										8	8					8	8		8
1	2	3	4	5	6	7	8	MESHES	1	1	1	0	0	0	0	0	0	0	0
	2		4	5	6	7			2	0	-1	1	1	0	0	0	0	0	0
			4						3	0	0	0	-1	1	1	0	0	0	0
		3			6	7	8		4	-1	0	0	0	0	0	1	1	0	0
				5		7			5	0	0	-1	0	0	0	0	-1	1	0
							8		6	0	0	0	0	0	0	-1	0	0	1

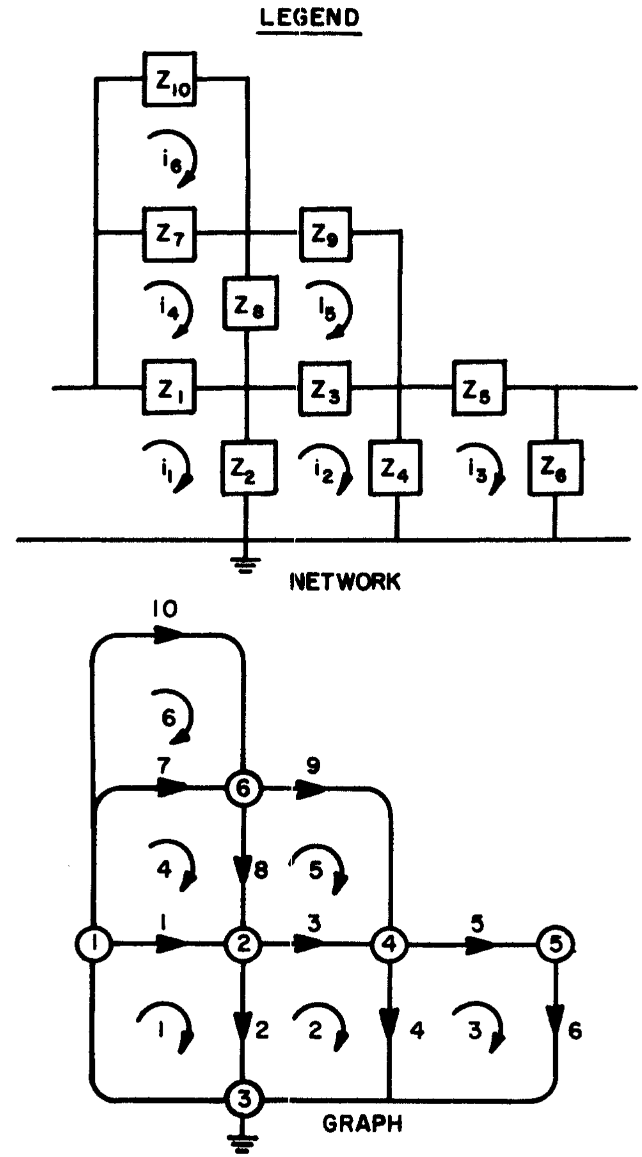
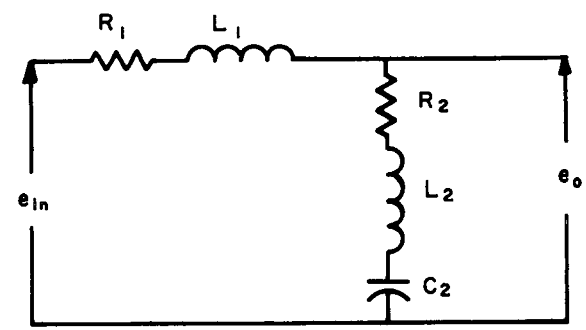


Figure IV-2. Generalized Network-Graph and Incidence Array



$$T(s) = \frac{N_s S^2 + N_1 S + N_o}{D_2 S^2 + D_1 S + D_o}$$

where

$$N_2 = L_2$$

$$N_1 = R_2$$

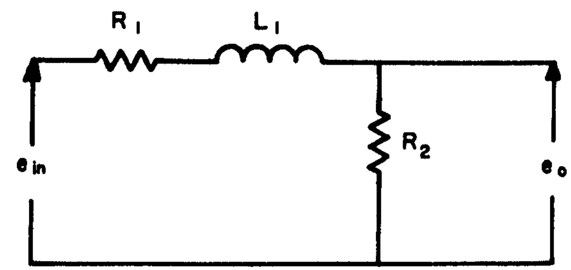
$$N_o = 1/C_2$$

$$D_2 = L_1 + L_2$$

$$D_1 = R_1 + R_2$$

$$D_o = 1/C_2$$

Figure IV-3. One Loop, 5-Element Circuit



$$T(s) = \frac{N_o}{D_1 S + D_o}$$

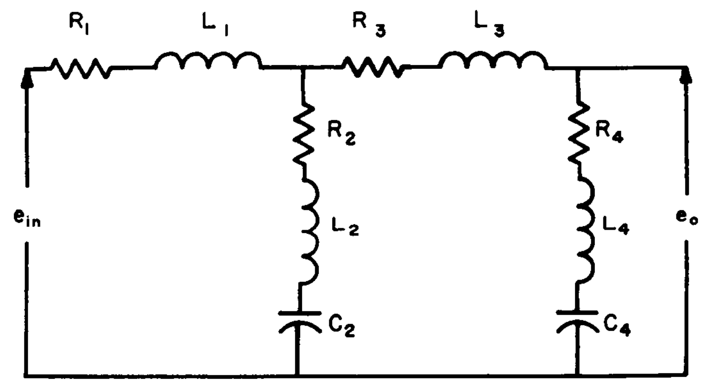
where

$$N_o = R_2$$

$$D_1 = L_1$$

$$D_o = R_1 + R_2$$

Figure IV-4. Resistive Load, One Loop, 3-Element Circuit

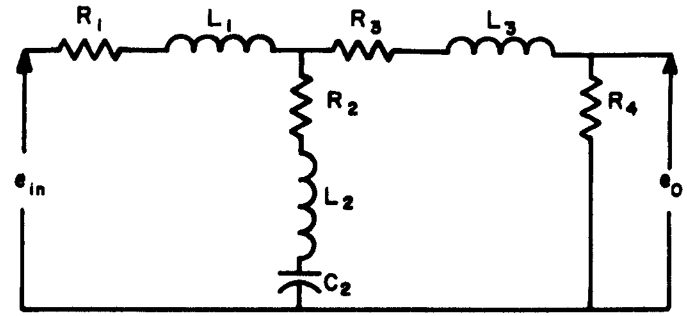


$$T(s) = \frac{N_4 S^4 + N_3 S^3 + N_2 S^2 + N_1 S + N_o}{D_4 S^4 + D_3 S^3 + D_2 S^2 + D_1 S + D_o}$$

where

$$N_4 = L_2 L_4$$
$$N_3 = R_2 L_4 + R_4 L_2$$
$$N_2 = R_2 R_4 + L_2/C_4 + L_4/C_2$$
$$N_1 = R_2/C_4 + R_4/C_2$$
$$N_o = 1/C_2 C_4$$
$$D_4 = L_2 L_4 + L_2 L_3 + L_1 L_4 + L_1 L_2 + L_1 L_3$$
$$D_3 = R_2 L_4 + R_4 L_2 + R_3 L_2 + R_2 L_3 + R_4 L_1 + R_1 L_4 + R_1 L_3 + R_3 L_1 + R_2 L_1 + R_1 L_2$$
$$D_2 = R_2 R_4 + L_2/C_4 + L_4/C_2 + R_3 R_2 + L_3/C_2 + R_1 R_4 + L_1/C_4 + R_1 R_3 + R_1 R_2 + L_1/C_2$$
$$D_1 = R_2/C_4 + R_4/C_2 + R_3/C_2 + R_1/C_4 + R_1/C_2$$
$$D_o = 1/C_2 C_4$$

Figure IV-5. Two Loop, 10-Element Circuit



$$T(s) = \frac{N_2 S^2 + N_1 S + N_o}{D_3 S^3 + D_2 S^2 + D_1 S + D_o}$$

where

$$N_2 = R_4 L_2$$

$$N_1 = R_2 R_4$$

$$N_o = R_4/C_2$$

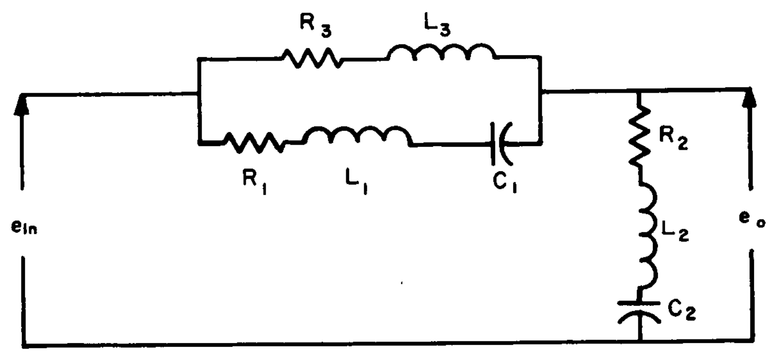
$$D_3 = L_2 L_3 + L_1 L_2 + L_1 L_3$$

$$D_2 = R_4 L_2 + R_3 L_2 + R_2 L_3 + R_4 L_1 + R_1 L_3 + R_3 L_1 + R_2 L_1 + R_1 L_2$$

$$D_1 = R_2 R_4 + R_3 R_2 + L_3/C_2 + R_1 R_4 + R_1 R_3 + R_1 R_2 + L_1/C_2$$

$$D_o = R_4/C_2 + R_3/C_2 + R_1/C_2$$

Figure IV-6. Resistive Load, Two Loops, 8-Element Circuit



$$T(s) = \frac{N_4 S^4 + N_3 S^3 + N_2 S^2 + N_1 S + N_o}{D_4 S^4 + D_3 S^3 + D_2 S^2 + D_1 S + D_o}$$

where

$$N_4 = L_2 L_3 + L_1 L_2$$

$$N_3 = R_2 L_3 + R_3 L_2 + R_2 L_2 + R_1 L_2$$

$$N_2 = R_2 R_3 + L_3/C_2 + R_1 R_2 + L_1/C_2 + L_2/C_1$$

$$N_1 = R_3/C_2 + R_2/C_1 + R_1/C_1$$

$$N_o = 1/C_1 C_2$$

$$D_4 = L_2 L_3 + L_1 L_2 + L_1 L_3$$

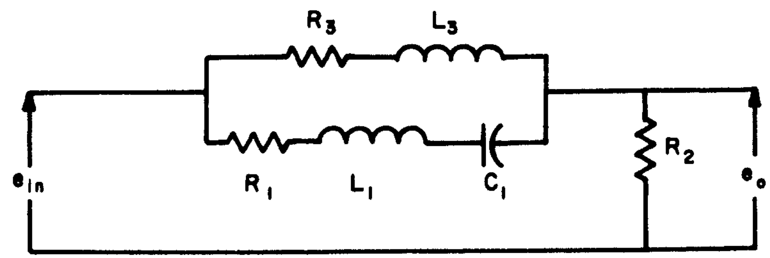
$$D_3 = R_2 L_3 + R_3 L_2 + R_2 L_1 + R_1 L_2 + R_1 L_3 + R_3 L_1$$

$$D_2 = R_2 R_3 + L_3/C_2 + R_1 R_2 + L_1/C_2 + L_2/C_1 + R_1 R_3 + L_3/C_1$$

$$D_1 = R_3/C_2 + R_2/C_1 + R_1/C_2 + R_3/C_1$$

$$D_o = 1/C_1 C_2$$

Figure IV-7. Two Loop Circuit, 8-Element Circuit

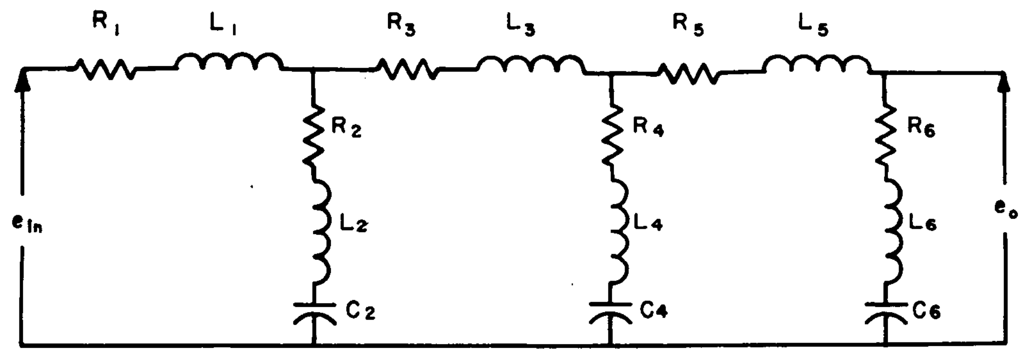


$$T(s) = \frac{N_2 S^2 + N_1 S + N_o}{D_3 S^3 + D_2 S^2 + D_1 S + D_o}$$

where

$$N_2 = R_2 L_3$$
$$N_1 = R_2 R_3 + R_1 R_2$$
$$N_o = R_2/C_1$$
$$D_3 = L_1 L_3$$
$$D_2 = R_2 L_3 + R_2 L_1 + R_1 L_3 + R_3 L_1$$
$$D_1 = R_2 R_3 + R_1 R_2 + R_1 R_3 + L_3/C_1$$
$$D_o = R_2/C_1 + R_3/C_1$$

Figure IV-8. Resistive Load, Two Loop, 6-Element Circuit



$$T(s) = \frac{N(s)}{D(s)} = \frac{N_6 S^6 + N_5 S^5 + N_4 S^4 + N_3 S^3 + N_2 S^2 + N_1 S + N_o}{D_6 S^6 + D_5 S^5 + D_4 S^4 + D_3 S^3 + D_2 S^2 + D_1 S + D_o}$$

where

$N_6 = L_2 L_4 L_6$

$N_5 = R_2 L_4 L_6 + R_4 L_2 L_6 + R_6 L_2 L_4$

$N_4 = R_2 R_4 L_6 + R_2 R_6 L_4 + R_4 R_6 L_2 + L_2 L_4/C_6 + L_2 L_6/C_4 + L_4 L_6/C_2$

$N_3 = R_2 R_4 R_6 + R_2 L_4/C_6 + R_2 L_6/C_4 + R_4 L_2/C_6 + R_4 L_6/C_2 + R_6 L_4/C_2 + R_6 L_2/C_4$

$N_2 = R_2 R_4/C_6 + R_2 R_6/C_4 + L_2/C_4 C_6 + R_4 R_6/C_2 + L_4/C_2 C_6 + L_6/C_2 C_6$

$N_1 = R_2/C_4 C_6 + R_4/C_2 C_6 + R_6/C_2 C_4$

$N_o = 1/C_2 C_4 C_6$

$D_6 = L_1 L_2 L_4 + L_1 L_3 L_4 + L_2 L_3 L_4 + L_1 L_2 L_5 + L_1 L_3 L_5 + L_2 L_3 L_5 + L_2 L_4 L_5 + L_1 L_3 L_6 +$
$L_1 L_4 L_6 + L_2 L_3 L_6 + L_2 L_4 L_6 + L_1 L_4 L_5 + L_1 L_2 L_6$

Figure IV-9. Three Loop, 15-Element Circuit

Continued:

$$
\begin{aligned}
D_5 = \; & R_2 L_4 L_6 + R_4 L_2 L_6 + R_6 L_2 L_4 + R_2 L_3 L_6 + R_3 L_2 L_6 + R_6 L_2 L_3 + R_1 L_4 L_6 + L_1 R_4 L_6 + \\
& L_1 L_4 R_6 + R_4 L_2 L_3 + R_1 L_3 L_6 + R_3 L_3 L_6 + L_1 L_3 R_6 + R_1 L_2 L_6 + R_2 L_1 L_6 + R_6 L_1 L_2 + \\
& R_2 L_4 L_5 + R_4 L_2 L_5 + R_5 L_2 L_4 + R_2 L_3 L_5 + R_3 L_2 L_5 + L_2 L_3 R_5 + R_1 L_4 L_5 + R_4 L_1 L_5 + \\
& R_5 L_1 L_4 + R_1 L_3 L_5 + R_3 L_1 L_5 + R_5 L_1 L_3 + R_1 L_2 L_5 + R_2 L_1 L_5 + R_5 L_1 L_2 + R_2 L_3 L_4 + \\
& R_3 L_2 L_4 + R_1 L_3 L_4 + R_3 L_1 L_4 + R_4 L_1 L_3 + R_1 L_2 L_4 + L_1 R_2 L_4 + R_4 L_1 L_2
\end{aligned}
$$

$$
\begin{aligned}
D_4 = \; & R_1 R_2 L_4 + R_1 R_4 L_2 + R_2 R_4 L_1 + L_1 L_2/C_4 + L_1 L_4/C_2 + R_1 R_3 L_4 + R_1 R_4 L_3 + R_3 R_4 L_1 + \\
& L_1 L_3/C_4 + R_2 R_3 L_4 + R_2 R_4 L_3 + L_2 R_3 R_4 + L_2 L_3/C_4 + L_3 L_4/C_2 + L_1 L_5/C_2 + R_1 R_5 L_2 + \\
& R_1 R_2 L_5 + L_1 R_2 R_5 + R_1 R_3 L_5 + R_1 R_5 L_3 + R_3 R_5 L_1 + R_1 R_4 L_5 + R_1 R_5 L_4 + R_4 R_5 L_1 + \\
& L_1 L_5/C_4 + R_2 R_3 L_5 + R_2 R_5 L_3 + L_2 R_3 R_5 + L_3 L_5/C_2 + R_2 R_4 L_5 + R_2 R_5 L_4 + R_4 R_5 L_2 + \\
& L_2 L_5/C_4 + L_4 L_5/C_2 + R_1 R_2 L_6 + R_1 R_6 L_2 + L_1 R_2 R_6 + L_1 L_2/C_6 + L_1 L_6/C_2 + R_1 R_3 L_6 + \\
& R_1 R_6 L_3 + R_3 R_6 L_1 + L_1 L_3/C_6 + R_1 R_4 L_6 + R_1 R_6 L_4 + L_1 R_4 R_6 + L_1 L_4/C_6 + L_1 L_6/C_4 + \\
& R_2 R_3 L_6 + R_2 R_6 L_3 + L_2 R_3 R_6 + L_2 L_3/C_6 + L_3 L_6/C_2 + R_2 R_4 L_6 + R_2 R_6 L_4 + R_4 R_6 L_2 + \\
& L_2 L_4/C_6 + L_2 L_6/C_4 + L_4 L_6/C_2
\end{aligned}
$$

$$
\begin{aligned}
D_3 = \; & R_2 R_4 R_6 + R_2 L_4/C_6 + R_2 L_6/C_4 + R_4 L_2/C_6 + R_6 L_2/C_4 + R_4 L_6/C_2 + R_6 L_4/C_2 + R_2 R_3 R_6 + \\
& R_2 L_3/C_6 + R_3 L_2/C_6 + R_3 L_6/C_2 + R_6 L_3/C_2 + R_1 R_4 R_6 + R_1 L_4/C_6 + R_4 L_1/C_6 + R_6 L_1/C_4 + \\
& R_1 L_6/C_4 + R_1 R_3 R_6 + R_1 L_3/C_6 + L_1 R_3/C_6 + R_1 R_2 R_6 + R_1 L_2/C_6 + R_1 L_6/C_2 + R_2 L_1/C_6 + \\
& R_6 L_1/C_2 + R_2 R_4 R_5 + R_4 L_5/C_2 + R_5 L_4/C_2 + R_2 L_5/C_4 + R_5 L_2/C_4 + R_2 R_3 R_5 + R_3 L_5/C_2 + \\
& R_5 L_3/C_2 + R_1 R_4 R_5 + R_1 L_5/C_4 + R_5 L_1/C_4 + R_1 R_3 R_5 + R_1 R_2 R_5 + R_1 L_5/C_2 + R_5 L_1/C_2 + \\
& R_2 R_3 R_4 + R_2 L_3/C_4 + R_3 L_2/C_4 + R_3 L_4/C_2 + R_4 L_3/C_2 + R_1 R_3 R_4 + R_1 L_3/C_4 + R_3 L_1/C_4 + \\
& R_1 R_2 R_4 + R_1 L_2/C_4 + R_1 L_4/C_2 + R_2 L_1/C_4 + R_4 L_1/C_2
\end{aligned}
$$

Figure IV-9. Three Loop, 15-Element Circuit

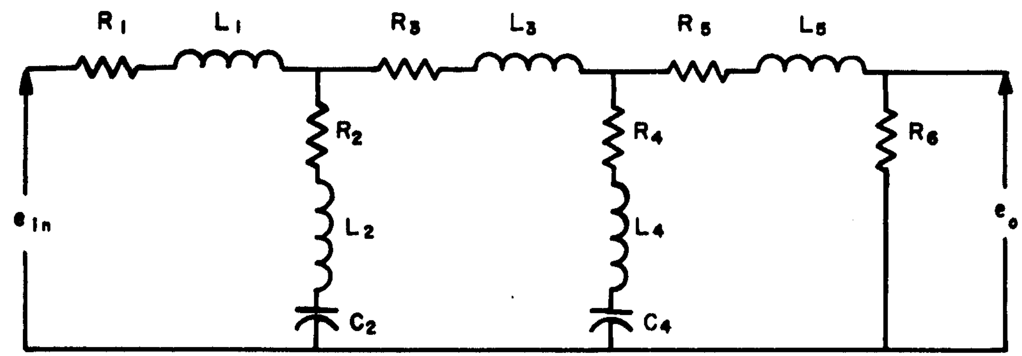
Continued:

$$D_2 = R_1 R_2/C_4 + R_1 R_4/C_2 + L_1/C_2C_4 + R_1 R_3/C_4 + R_2 R_3/C_4 + R_3 R_4/C_2 + L_3/C_2 C_6 + R_2 R_4/C_6 +$$
$$R_2 R_6/C_4 + L_2/C_4C_6 + R_4 R_6/C_2 + L_4/C_2C_6 + L_6/C_2C_4 + L_3/C_2 C_4 + R_1 R_5/C_2 + R_1 R_5/C_4 +$$
$$R_3 R_5/C_2 + R_4 R_5/C_2 + R_2 R_5/C_4 + L_5/C_2 C_4 + R_1 R_2/C_6 + R_1 R_6/C_2 + L_1/C_2C_6 + R_1 R_3/C_6 +$$
$$R_1 R_4/C_6 + R_1 R_6/C_4 + L_1/C_4C_6 + R_2 R_3/C_6 + R_3 R_6/C_2$$

$$D_1 = R_2/C_4 C_6 + R_4/C_2C_6 + R_6/C_2C_4 + R_3/C_2/C_6 + R_1/C_4 C_6 + R_1/C_2C_6 + R_5/C_2C_4 + R_3/ C_2 C_4 +$$
$$R_1/C_2 C_4$$

$$D_o = 1/C_2 C_4 C_6$$

Figure IV-9. Three Loop, 15-Element Circuit



$$T(s) = \frac{N_4 S^4 + N_3 S^3 + N_2 S^2 + N_1 S + N_o}{D_5 S^5 + D_4 S^4 + D_3 S^3 + D_2 S^2 + D_1 S + D_o}$$

where

$N_4 = R_6 L_2 L_4$

$N_3 = R_2 R_6 L_4 + R_4 R_6 L_2$

$N_2 = R_2 R_4 R_6 + R_6 L_2/C_4 + R_6 L_4/C_2$

$N_1 = R_2 R_6/C_4 + R_4 R_6/C_2$

$N_o = R_6 / C_2 C_4$

$D_5 = L_1 L_2 L_4 + L_1 L_3 L_4 + L_2 L_3 L_4 + L_1 L_2 L_5 + L_1 L_3 L_5 + L_1 L_4 L_5 + L_2 L_3 L_5 + L_2 L_4 L_5$

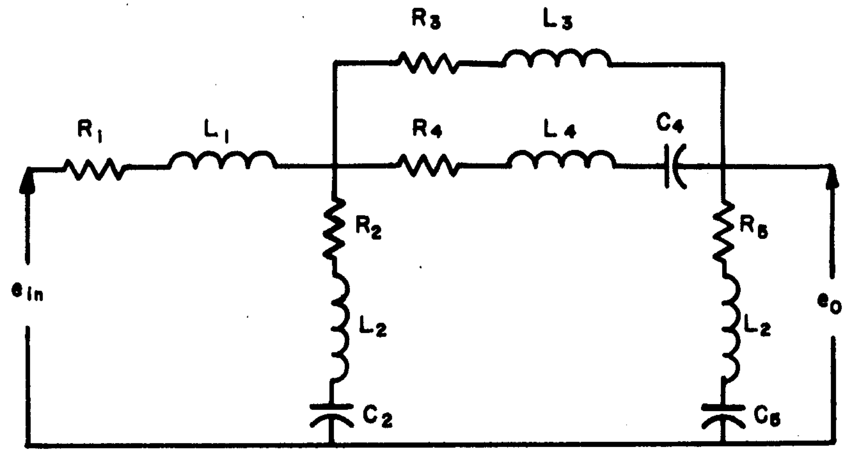
$D_4 = R_6 L_2 L_4 + R_6 L_2 L_3 + R_6 L_1 L_4 + R_6 L_1 L_3 + R_6 L_1 L_2 + R_2 L_4 L_5 + R_4 L_2 L_5 + R_5 L_2 L_4 +$
$R_2 L_3 L_5 + R_3 L_2 L_5 + R_5 L_2 L_3 + R_1 L_4 L_5 + R_4 L_1 L_5 + R_5 L_1 L_4 + R_1 L_3 L_5 + R_3 L_1 L_5 +$
$R_5 L_1 L_3 + R_1 L_2 L_5 + R_2 L_1 L_5 + R_5 L_1 L_2 + R_2 L_3 L_4 + R_3 L_2 L_4 + R_4 L_2 L_3 + R_1 L_3 L_4 +$
$R_3 L_1 L_4 + R_4 L_1 L_3 + R_1 L_2 L_4 + R_2 L_1 L_4 + R_4 L_1 L_2$

Figure IV-10. Resistive Load, Three Loop, 13-Element Circuit

Continued:

$$
\begin{aligned}
D_3 = {} & R_1 R_2 L_4 + R_1 R_4 L_2 + R_2 R_4 L_1 + L_1 L_2/C_4 + L_1 L_4/C_2 + R_1 R_3 L_4 + R_1 R_4 L_3 + R_3 R_4 L_1 + \\
& L_1 L_3/C_4 + R_2 R_3 L_4 + R_2 R_4 L_3 + R_3 R_4 L_2 + L_2 L_3/C_4 + L_3 L_4/C_2 + L_1 L_5/C_2 + R_1 R_5 L_2 + \\
& R_1 R_2 L_5 + R_2 R_5 L_1 + R_1 R_3 L_5 + R_1 R_5 L_3 + R_3 R_5 L_1 + R_1 R_4 L_5 + R_1 R_5 L_4 + R_4 R_5 L_1 + \\
& L_1 L_5/C_4 + R_2 R_3 L_5 + R_2 R_5 L_3 + R_3 R_5 L_2 + L_3 L_5/C_2 + R_2 R_4 L_5 + R_2 R_5 L_4 + R_4 R_5 L_2 + \\
& L_2 L_5/C_4 + L_4 L_5/C_2 + R_1 R_6 L_2 + R_2 R_6 L_1 + R_1 R_6 L_3 + R_3 R_6 L_1 + R_1 R_6 L_4 + R_4 R_6 L_1 + \\
& R_2 R_6 L_3 + R_3 R_6 L_2 + R_2 R_6 L_4 + R_4 R_6 L_2 \\
D_2 = {} & R_2 R_4 R_6 + R_6 L_2/C_4 + R_6 L_4/C_2 + R_2 R_3 R_6 + R_6 L_3/C_2 + R_1 R_4 R_6 + R_6 L_1/C_4 + R_1 R_3 R_6 + \\
& R_1 R_2 R_6 + R_6 L_1/C_2 + R_2 R_4 R_5 + R_4 L_5/C_2 + R_5 L_4/C_2 + R_2 L_5/C_4 + R_5 L_2/C_4 + R_2 R_3 R_5 + \\
& R_3 L_5/C_2 + R_5 L_3/C_2 + R_1 R_4 R_5 + R_1 L_5/C_4 + R_5 L_1/C_4 + R_1 R_3 R_5 + R_1 R_2 R_5 + R_1 L_5/C_2 + \\
& R_5 L_1/C_2 + R_2 R_3 R_4 + R_2 L_3/C_4 + R_3 L_2/C_4 + R_3 L_4/C_2 + R_4 L_3/C_2 + R_1 R_3 R_4 + R_1 L_3/C_4 + \\
& R_3 L_1/C_4 + R_1 R_2 R_4 + R_1 L_2/C_4 + R_1 L_4/C_2 + R_2 L_1/C_4 + R_4 L_1/C_2 \\
D_1 = {} & R_1 R_2/C_4 + R_1 R_4/C_2 + L_1/C_2 C_4 + R_1 R_3/C_4 + R_2 R_3/C_4 + R_3 R_4/C_2 + R_2 R_6/C_4 + R_4 R_6/C_2 + \\
& L_3/C_2 C_4 + R_1 R_5/C_2 + R_1 R_5/C_4 + R_3 R_5/C_2 + R_4 R_5/C_2 + R_2 R_5/C_4 + L_5/C_2 C_4 + R_1 R_6/C_2 + \\
& R_1 R_6/C_4 + R_3 R_6/C_2 \\
D_o = {} & R_6/C_2 C_4 + R_5/C_2 C_4 + R_3/C_2 C_4 + R_1/C_2 C_4
\end{aligned}
$$

Figure IV-10. Resistive Load, Three Loop, 13-Element Circuit



$$T(s) = \frac{N_6 S^6 + N_5 S^5 + N_4 S^4 + N_3 S^3 + N_2 S^2 + N_1 S + N_o}{D_6 S^6 + D_5 S^5 + D_4 S^4 + D_3 S^3 + D_2 S^2 + D_1 S + D_o}$$

where

$$N_6 = L_2 L_3 L_5 + L_2 L_4 L_5$$

$$N_5 = R_2 L_3 L_5 + R_2 L_4 L_5 + R_5 L_2 L_3 + R_5 L_2 L_4 + R_3 L_2 L_5 + R_4 L_2 L_5$$

$$N_4 = R_2 R_5 L_3 + R_2 R_5 L_4 + R_2 R_3 L_5 + R_2 R_4 L_5 + R_3 R_5 L_2 + R_4 R_5 L_2 + L_2 L_5/C_4 + L_3 L_5/C_2 + L_4 L_5/C_2 + L_2 L_3/C_5 + L_2 L_4/C_5$$

$$N_3 = R_2 R_3 R_5 + R_2 R_4 R_5 + R_2 L_5/C_4 + R_2 L_3/C_5 + R_2 L_4/C_5 + R_5 L_2/C_4 + R_3 L_2/C_5 + R_4 L_2/C_5 + R_5 L_3/C_2 + R_5 L_4/C_2 + R_3 L_5/C_2 + R_4 L_5/C_2$$

$$N_2 = R_2 R_5/C_4 + R_2 R_3/C_5 + R_3 R_5/C_2 + R_4 R_5/C_2 + L_5/C_2 C_4 + L_3/C_2 C_5 + L_4/C_2 C_5 + R_2 R_4/C_5 + L_2/C_4 C_5$$

$$N_1 = R_2/C_4 C_5 + R_5/C_2 C_4 + R_3/C_2 C_5 + R_4/C_2 C_5$$

$$N_o = 1/C_2 C_4 C_5$$

Figure IV-11. Three Loop 13-Element Circuit

Continued:

$$D_6 = L_1L_2L_3 + L_1L_3L_4 + L_1L_3L_5 + L_1L_2L_4 + L_1L_4L_5 + L_2L_3L_4 + L_2L_3L_5 + L_2L_3L_5 + L_2L_4L_5$$

$$D_5 = R_1L_2L_3 + R_3L_1L_2 + R_2L_1L_3 + R_1L_3L_4 + R_3L_1L_4 + R_4L_1L_3 + R_1L_3L_5 + R_3L_1L_5 + R_5L_1L_3 + R_1L_2L_4 + R_4L_1L_2 + R_2L_1L_4 + R_1L_4L_5 + R_4L_1L_5 + R_5L_1L_4 + R_2L_3L_4 + R_3L_2L_4 + R_4L_2L_3 + R_2L_3L_5 + R_3L_2L_5 + R_5L_2L_3 + R_2L_4L_5 + R_4L_2L_5 + R_5L_2L_4$$

$$D_4 = R_1R_3L_2 + R_1R_2L_3 + R_2R_3L_1 + L_1L_3/C_2 + R_1R_3L_4 + R_1R_4L_3 + R_3R_4L_1 + L_1L_3/C_4 + R_1R_3L_5 + R_1R_5L_3 + R_3R_5L_1 + L_1L_3/C_5 + R_1R_4L_2 + R_1R_2L_4 + R_2R_4L_1 + L_1L_4/C_2 + L_1L_2/C_4 + R_1R_4L_5 + R_1R_5L_4 + R_4R_5L_1 + L_1L_4/C_5 + L_1L_5/C_4 + R_2R_3L_4 + R_2R_4L_3 + R_3R_4L_2 + L_2L_3/C_4 + L_3L_4/C_2 + R_2R_3L_5 + R_2R_5L_3 + R_3R_5L_2 + L_2L_3/C_4 + L_3L_5/C_2 + R_2R_4L_5 + R_2R_5L_4 + R_4R_5L_2 + L_2L_4/C_5 + L_2L_5/C_4 + L_4L_5/C_2$$

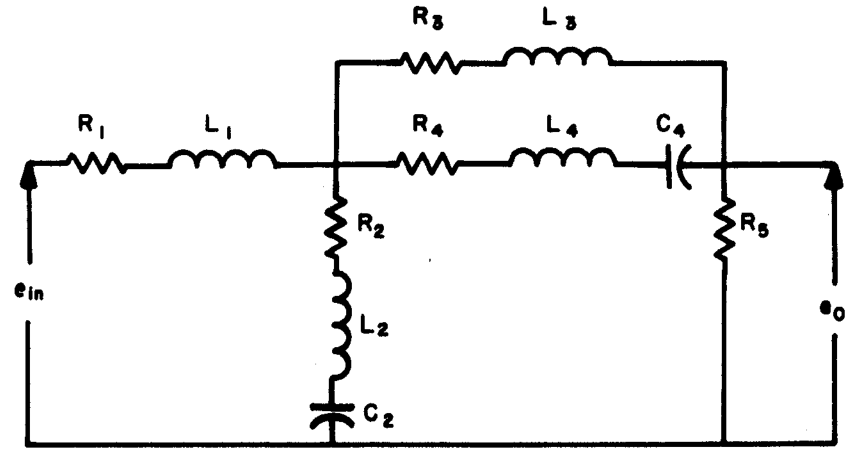
$$D_3 = R_1R_3R_2 + R_1L_3/C_2 + R_3L_1/C_2 + R_1R_3R_4 + R_1L_3/C_4 + R_3L_1/C_4 + R_1R_3R_5 + R_1L_3/C_5 + R_3L_1/C_5 + R_1R_4R_2 + R_1L_4/C_2 + R_1L_2/C_4 + R_4L_1/C_2 + R_2L_1/C_4 + R_1R_4R_5 + R_1L_4/C_5 + R_1L_5/C_4 + R_4L_1/C_5 + R_5L_1/C_4 + R_2R_3R_4 + R_2L_3/C_4 + R_3L_2/C_4 + R_3L_4 \cdot C_2 + R_4L_3/C_2 + R_2R_3R_5 + R_2L_3/C_5 + R_3L_2/C_5 + R_3L_5/C_2 + R_5L_3/C_2 + R_2R_4R_5 + R_2L_4/C_5 + R_2L_5/C_4 + R_4L_2/C_5 + R_5L_2/C_4 + R_4L_5/C_2 + R_5L_4/C_2$$

$$D_2 = R_1R_3/C_2 + R_1R_3/C_4 + R_1R_3/C_5 + R_1R_4/C_2 + R_1R_2/C_4 + L_1/C_2C_4 + R_1R_4/C_5 + R_1R_5/C_4 + L_1/C_4C_5 + R_2R_3/C_4 + R_3R_4/C_2 + L_3/C_2C_4 + R_2R_3/C_5 + R_3R_5/C_2 + L_3/C_2C_5 + R_2R_4/C_5 + R_2R_5/C_4 + L_2/C_4C_5 + R_4R_5/C_2 + L_4/C_2C_5 + L_5/C_2C_4$$

$$D_1 = R_1/C_2C_4 + R_1/C_4C_5 + R_3/C_2C_4 + R_3/C_2C_5 + R_2/C_4C_5 + R_4/C_2C_5 + R_5/C_2C_4$$

$$D_o = 1/C_2C_4C_5$$

Figure IV-11 Three Loop, 13-Element Circuit



$$T(s) = \frac{N_4S^4 + N_3S^3 + N_2S^2 + N_1S + N_o}{D_5S^5 + D_4S^4 + D_3S^3 + D_2S^2 + D_1S + D_o}$$

where

$$N_4 = R_5L_2L_3 + R_5L_2L_4$$

$$N_3 = R_2R_5L_3 + R_2R_5L_4 + R_3R_5L_2 + R_4R_5L_2$$

$$N_2 = R_2R_3R_5 + R_2R_4R_5 + R_5L_2/C_4 + R_5L_3/C_2 + R_5L_4/C_2$$

$$N_1 = R_2R_5/C_4 + R_3R_5/C_2 + R_4R_5/C_2$$

$$N_o = R_5/C_2C_4$$

$$D_5 = L_1L_2L_3 + L_1L_3L_4 + L_1L_2L_4 + L_2L_3L_4$$

$$D_4 = R_1L_2L_3 + R_3L_1L_2 + R_2L_1L_3 + R_1L_3L_4 + R_3L_1L_4 + R_4L_1L_3 + R_5L_1L_3 + R_1L_2L_4 + R_4L_1L_2 + R_2L_1L_4 +$$
$$R_5L_1L_4 + R_2L_3L_4 + R_3L_2L_4 + R_4L_2L_3 + R_5L_2L_3 + R_5L_2L_4$$

$$D_3 = R_1R_3L_2 + R_1R_2L_3 + R_2R_3L_1 + L_1L_3/C_2 + R_1R_3L_4 + R_1R_4L_3 + R_3R_4L_1 + L_1L_3/C_4 + R_1R_5L_3 + R_3R_5L_1 +$$

Figure IV-12. Resistive Load, Three Loop, 11-Element Circuit

Continued:

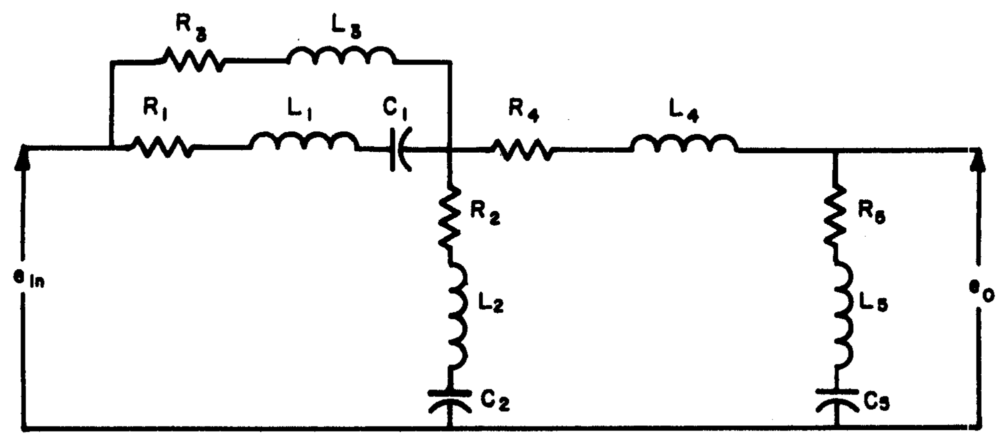
$$R_1R_4L_2 + R_1R_2L_4 + R_2R_4L_1 + L_1L_4/C_2 + L_1L_2/C_4 + R_1R_5L_4 + R_4R_5L_1 + R_2R_3L_4 + R_2R_4L_3 +$$
$$R_3R_4L_2 + L_2L_3/C_4 + L_3L_4/C_2 + R_2R_5L_3 + R_3R_5L_2 + L_2L_3/C_4 + R_2R_5L_4 + R_4R_5L_2$$

$$D_2 = R_1R_2R_3 + R_1L_3/C_2 + R_3L_1/C_2 + R_1R_3R_4 + R_1L_3/C_4 + R_3L_1/C_4 + R_1R_3R_5 + R_1R_2R_4 + R_1L_4/C_2 +$$
$$R_1L_2/C_4 + R_4L_1/C_2 + R_2L_1/C_4 + R_1R_4R_5 + R_5L_1/C_4 + R_2R_3R_4 + R_2L_3/C_4 + R_3L_2/C_4 + R_3L_4/C_2 +$$
$$R_4L_3/C_2 + R_2R_3R_5 + R_5L_3/C_2 + R_2R_4R_5 + R_5L_2/C_4 + R_5L_4/C_2$$

$$D_1 = R_1R_3/C_2 + R_1R_3/C_4 + R_1R_4/C_2 + R_1R_2/C_4 + L_1/C_2C_4 + R_1R_5/C_4 + R_2R_3/C_4 + R_3R_4/C_2 + L_3/C_2C_4 +$$
$$R_3R_5/C_2 + R_2R_5/C_4 + R_4R_5/C_2 + L_5/C_2C_4$$

$$D_o = R_1/C_2C_4 + R_3/C_2C_4 + R_5/C_2C_4$$

Figure IV-12. Resistive Load, Three Loop, 11-Element Circuit



$$T(s) = \frac{N_6S^6 + N_5S^5 + N_4S^4 + N_3S^3 + N_2S^2 + N_1S + N_o}{D_6S^6 + D_5S^5 + D_4S^4 + D_cS^3 + D_2S^2 + D_1S + D_o}$$

where

$$N_6 = L_1L_3L_5 + L_1L_2L_5$$

$$N_5 = R_2L_3L_5 + R_3L_2L_5 + R_5L_2L_3 + R_1L_2L_5 + R_2L_1L_5 + R_5L_1L_2$$

$$N_4 = R_2R_5L_3 + R_2R_3L_5 + R_3R_5L_2 + L_2L_3/C_5 + L_3L_5/C_2 + R_1R_2L_5 + R_1R_5L_2 + R_2R_5L_1 + L_1L_2/C_5 + L_1L_5/C_2 + L_2L_5/C_1$$

$$N_3 = R_2R_3R_5 + R_2L_3/C_5 + R_3L_2/C_5 + R_3L_5/C_2 + R_5L_3/C_2 + R_1R_2R_5 + R_1L_2/C_5 + R_1L_5/C_2 + R_2L_1/C_5 + R_5L_1/C_2 + R_2L_5/C_1 + R_5L_2/C_1$$

$$N_2 = R_2R_3/C_5 + R_3R_5/C_2 + L_3/C_2C_5 + R_1R_2/C_5 + R_1R_5/C_2 + L_1/C_2C_5 + R_2R_5/C_1 + L_2/C_1C_5 + L_5/C_1C_2$$

$$N_1 = R_3/C_2C_5 + R_1/C_2C_5 + R_2/C_1C_5 + R_5/C_1C_2$$

$$N_o = 1/C_1C_2C_5$$

Figure IV-13. Three Loop, 13-Element Circuit

Continued:

$$D_6 = L_1L_2L_3 + L_1L_2L_4 + L_1L_3L_5 + L_1L_3L_4 + L_1L_2L_5 + L_2L_3L_5 + L_2L_3L_4$$

$$\begin{aligned} D_5 = {} & R_1L_2L_3 + R_2L_1L_3 + R_3L_1L_2 + R_1L_2L_4 + R_2L_1L_4 + R_4L_1L_2 + R_1L_3L_4 + R_3L_1L_4 + R_4L_1L_3 + \\ & R_1L_3L_5 + R_3L_1L_5 + R_5L_1L_3 + R_1L_2L_5 + R_2L_1L_5 + R_5L_1L_2 + R_2L_3L_5 + R_3L_2L_5 + R_5L_2L_3 + \\ & R_2L_3L_4 + R_3L_2L_4 + R_4L_2L_3 \end{aligned}$$

$$\begin{aligned} D_4 = {} & R_1R_2L_3 + R_1R_3L_2 + R_2R_3L_1 + L_1L_3/C_2 + L_2L_3/C_1 + R_1R_2L_4 + R_1R_4L_2 + R_2R_4L_1 + L_1L_4/C_2 + \\ & L_2L_4/C_1 + R_1R_5L_3 + R_1R_3L_5 + R_3R_5L_1 + L_1L_3/C_5 + L_3L_5/C_1 + R_1R_3L_4 + R_1R_4L_3 + R_3R_4L_1 + \\ & L_3L_4/C_1 + R_1R_2L_5 + R_1R_5L_2 + R_2R_5L_1 + L_1L_2/C_5 + L_1L_5/C_2 + L_2L_5/C_1 + R_2R_5L_3 + R_2R_3L_5 + \\ & R_3R_5L_2 + L_2L_3/C_5 + L_3L_5/C_2 + R_2R_3L_4 + R_2R_4L_3 + R_3R_4L_2 + L_3L_4/C_2 \end{aligned}$$

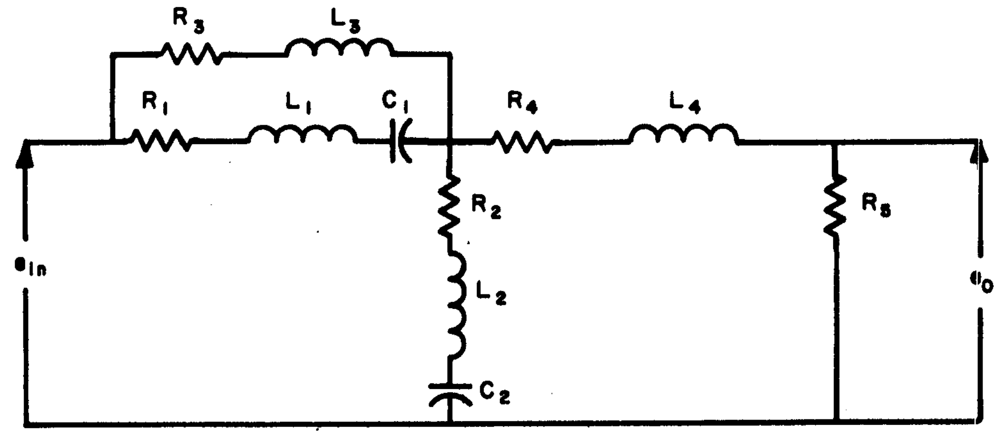
$$\begin{aligned} D_3 = {} & R_1R_2R_3 + R_1L_3/C_2 + R_3L_1/C_2 + R_2L_3/C_1 + R_3L_2/C_1 + R_1R_2R_4 + R_1L_4/C_2 + R_4L_1/C_2 + R_2L_4/C_1 + \\ & R_4L_2/C_1 + R_1R_3R_5 + R_1L_3/C_5 + R_3L_1/C_5 + R_3L_5/C_1 + R_5L_3/C_1 + R_1R_3R_4 + R_3L_4/C_1 + R_4L_3/C_1 + \\ & R_1R_2R_5 + R_1L_2/C_5 + R_2L_1/C_5 + R_5L_1/C_2 + R_2L_5/C_1 + R_5L_2/C_1 + R_2R_3R_5 + R_2L_3/C_5 + R_3L_2/C_5 + \\ & R_3L_5/C_2 + R_5L_3/C_2 + R_2R_3R_4 + R_3L_4/C_2 + R_4L_3/C_2 + R_1L_5/C_2 \end{aligned}$$

$$\begin{aligned} D_2 = {} & R_1R_3/C_2 + R_2R_3/C_1 + L_3/C_1C_2 + R_1R_4/C_2 + R_2R_4/C_1 + L_4/C_1C_2 + R_1R_3/C_5 + R_3R_5/C_1 + L_3/C_1C_5 + \\ & R_3R_4/C_1 + R_1R_2/C_5 + R_1R_5/C_2 + L_1/C_2C_5 + R_2R_5/C_1 + L_2/C_1C_5 + L_5/C_1C_2 + R_2R_3/C_5 + R_3R_5/C_2 + \\ & L_3/C_2C_5 + R_3R_4/C_2 \end{aligned}$$

$$D_1 = R_3/C_1C_2 + R_4/C_1C_2 + R_3/C_1C_5 + R_1/C_2C_5 + R_2/C_1C_5 + R_5/C_1C_2 + R_3/C_2C_5$$

$$D_o = 1/C_1C_2C_5$$

Figure IV-13. Three Loop, 13-Element Circuit



$$T(s) = \frac{N_4 S^4 + N_3 S^3 + N_2 S^2 + N_1 S + N_o}{D_5 S^5 + D_4 S^4 + D_3 S^3 + D_2 S^2 + D_1 S^1 + D_o}$$

where

$$N_4 = R_5 L_2 L_3 + R_5 L_1 L_2$$

$$N_3 = R_2 R_5 L_3 + R_3 R_5 L_2 + R_1 R_5 L_2 + R_2 R_5 L_1$$

$$N_2 = R_2 R_3 R_5 + R_5 L_3/C_2 + R_1 R_2 R_5 + R_5 L_1/C_2 + R_5 L_2/C_1$$

$$N_1 = R_3 R_5/C_2 + R_1 R_5/C_2 + R_2 R_5/C_1$$

$$N_o = R_5/C_1 C_2$$

$$D_5 = L_1 L_2 L_3 + L_1 L_2 L_4 + L_1 L_3 L_4 + L_2 L_3 L_4$$

$$D_4 = R_1 L_2 L_3 + R_2 L_1 L_3 + R_3 L_1 L_2 + R_1 L_2 L_4 + R_2 L_1 L_4 + R_4 L_1 L_2 + R_1 L_3 L_4 + R_3 L_1 L_4 + R_4 L_1 L_3 + R_5 L_1 L_3 + R_5 L_1 L_2 + R_5 L_2 L_3 + R_2 L_3 L_4 + R_3 L_2 L_4 + R_4 L_2 L_3$$

$$D_3 = R_1 R_2 L_3 + R_1 R_3 L_2 + R_2 R_3 L_1 + L_1 L_3/C_2 + L_2 L_3/C_1 + R_1 R_2 L_4 + R_1 R_4 L_2 + R_2 R_4 L_1 + L_1 L_4/C_2 + L_2 L_4/C_1 +$$

Figure IV-14. Resistive Load, Three Loop, 11-Element Circuit

Continued:

$$R_1R_5L_3 + R_3R_5L_1 + R_1R_3L_4 + R_1R_4L_3 + R_3R_4L_1 + L_3L_4/C_1 + R_1R_5L_2 + R_2R_5L_1 + R_2R_5L_3 +$$

$$R_3R_5L_2 + R_2R_3L_4 + R_2R_4L_3 + R_3R_4L_2 + L_3L_4/C_2$$

$$D_2 = R_1R_2R_3 + R_1L_3/C_2 + R_3L_1/C_2 + R_2L_3/C_1 + R_3L_2/C_1 + R_1R_2R_4 + R_1L_4/C_2 + R_4L_1/C_2 + R_2L_4/C_1 +$$

$$R_4L_2/C_1 + R_1R_3R_5 + R_5L_3/C_1 + R_1R_3R_4 + R_3L_4/C_1 + R_4L_3/C_1 + R_1R_2R_5 + R_1L_1/C_2 + R_2L_5/C_1 +$$

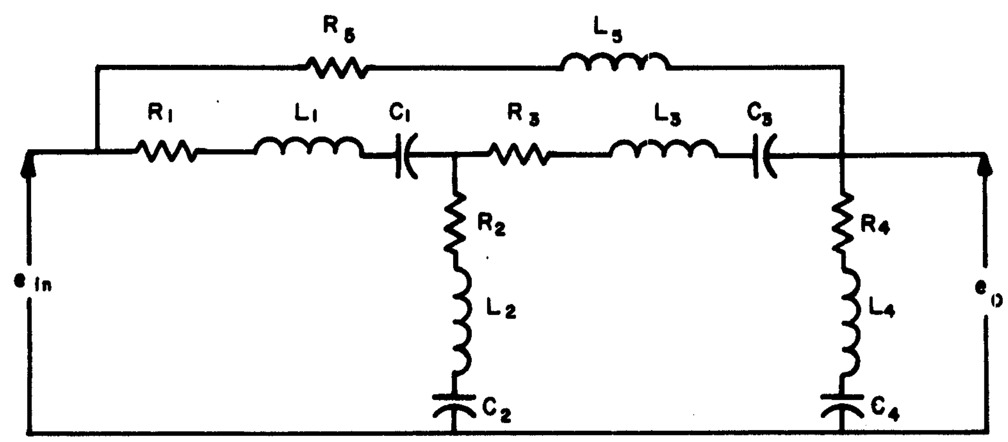
$$R_5L_2/C_1 + R_2R_3R_5 + R_5L_3/C_2 + R_2R_3R_4 + R_3L_4/C_2 + R_4L_3/C_2$$

$$D_1 = R_1R_3/C_2 + R_2R_3/C_1 + L_3/C_1C_2 + R_1R_4/C_2 + R_2R_4/C_1 + L_4/C_1C_2 + R_3R_5/C_1 + R_3R_4/C_1 + R_1R_5/C_2 +$$

$$R_2R_5/C_1 + R_3R_5/C_2 + R_3R_4/C_2$$

$$D_o = R_3/C_1C_2 + R_4/C_1C_2 + R_5/C_1C_2$$

Figure IV-14. Resistive Load, Three Loop, 11-Element Circuit



$$T(s) = \frac{N_6S^6 + N_5S^5 + N_4S^4 + N_3S^3 + N_2S^2 + N_1S + N_o}{D_6S^6 + D_5S^5 + D_4S^4 + D_3S^3 + D_2S^2 + D_1S + D_o}$$

where

$$N_6 = L_1L_3L_4 + L_1L_2L_4 + L_2L_4L_5 + L_2L_3L_4$$

$$N_5 = R_4L_1L_3 + R_3L_1L_4 + R_1L_3L_4 + R_4L_1L_2 + R_2L_1L_4 + R_1L_2L_4 + R_5L_2L_4 + R_4L_2L_5 + R_2L_4L_5 + R_4L_2L_3 + R_3L_2L_4 + R_2L_3L_4$$

$$N_4 = R_4R_3L_1 + R_4R_1L_3 + R_1R_3L_4 + L_1L_4/C_3 + L_3L_4/C_1 + L_1L_3/C_4 + R_4R_2L_1 + R_4R_1L_2 + R_1R_2L_4 + L_1L_4/C_2 + L_2L_4/C_1 + L_1L_2/C_4 + R_4R_5L_2 + R_2R_5L_4 + R_2R_4L_5 + L_2L_5/C_4 + L_4L_5/C_2 + R_3R_4L_2 + R_4R_2L_3 + R_2R_3L_4 + L_2L_4/C_3 + L_3L_4/C_2 + L_2L_3/C_4$$

$$N_3 = R_1R_3R_4 + R_4L_1/C_3 + R_4L_3/C_1 + R_3L_4/C_1 + R_1L_4/C_3 + R_3L_1/C_4 + R_1L_3/C_4 + R_1R_2R_4 + R_4L_1/C_2 +$$

Figure IV-15a. Three Loop, 14-Element Circuit

Continued:

$$R_4L_2/C_1 + R_2L_4/C_1 + R_1L_4/C_2 + R_2L_1/C_4 + R_1L_2/C_4 + R_2R_4R_5 + R_5L_2/C_4 + R_2L_5/C_4 + R_5L_4/C_2 +$$
$$R_4L_5/C_2 + R_2R_3R_4 + R_4L_2/C_3 + R_4L_3/C_2 + R_3L_4/C_2 + R_2L_4/C_3 + R_3L_2/C_4 + R_2L_3/C_4$$

$$N_2 = R_4R_3/C_1 + R_4R_1/C_3 + R_1R_2/C_4 + L_1/C_3C_4 + L_3/C_1C_4 + L_4/C_1C_3 + R_4R_2/C_1 + R_1R_4/C_2 + R_1R_2/C_4 +$$
$$L_1/C_2C_4 + L_2/C_1C_4 + L_4/C_1C_2 + R_5R_4/C_2 + R_2R_5/C_4 + L_5/C_2C_4 + R_3R_4/C_2 + R_2R_4/C_3 + R_2R_3/C_4 +$$
$$L_2/C_3C_4 + L_3/C_2C_4 + L_4/C_2C_3$$

$$N_1 = R_4/C_1C_3 + R_3/C_1C_4 + R_1/C_3C_4 + R_4/C_1C_2 + R_2/C_1C_4 + R_1/C_2C_4 + R_5/C_2C_4 +$$
$$R_4/C_2C_3 + R_3/C_2C_4 + R_2/C_3C_4$$

$$N_o = 1/C_1C_3C_4 + 1/C_1C_2C_4 + 1/C_2C_3C_4$$

$$D_6 = L_1L_3L_4 + L_1L_2L_4 + L_2L_4L_5 + L_2L_3L_4 + L_1L_3L_5 + L_2L_1L_5 + L_1L_4L_5 + L_2L_3L_5$$

$$D_5 = R_4L_1L_3 + R_3L_1L_4 + R_1L_3L_4 + R_4L_1L_2 + R_2L_1L_4 + R_1L_2L_4 + R_5L_2L_4 + R_4L_2L_5 + R_2L_4L_5 +$$
$$R_4L_2L_3 + R_3L_2L_4 + R_2L_3L_4 + R_5L_1L_4 + R_4L_1L_5 + R_1L_4L_5 + R_5L_2L_3 + R_3L_2L_5 + R_2L_3L_5 +$$
$$R_5L_1L_3 + R_3L_1L_5 + R_1L_3L_5 + R_5L_1L_2 + R_2L_1L_5 + R_1L_2L_5$$

$$D_4 = R_4R_3L_1 + R_4L_1L_3 + R_1R_3L_4 + L_1L_4/C_3 + L_3L_4/C_1 + L_1L_3/C_4 + R_4R_2L_1 + R_1R_4L_2 + R_1R_2L_4 +$$
$$L_1L_4/C_2 + L_2L_4/C_1 + L_1L_2/C_4 + R_5R_4L_2 + R_2R_5L_4 + R_2R_4L_5 + L_2L_5/C_4 + L_4L_5/C_2 + R_4R_3L_2 +$$
$$R_2R_4L_3 + R_2R_3L_4 + L_2L_4/C_3 + L_3L_4/C_2 + L_2L_3/C_4 + R_4R_5L_1 + R_1R_5L_4 + R_1R_4L_5 + L_1L_5/C_4 +$$
$$L_4L_5/C_1 + R_3R_5L_2 + R_2R_5L_3 + R_2R_3L_5 + L_2L_5/C_3 + L_3L_5/C_2 + R_3R_5L_1 + R_1R_5L_3 + R_1R_3L_5$$

Figure IV-15a. Three Loop, 14-Element Circuit

Continued:

$$L_1L_5/C_3 + L_3L_5/C_1 + R_2R_5L_1 + R_5R_1L_2 + R_1R_2L_5 + L_1L_5/C_2 + L_2L_5/C_1$$

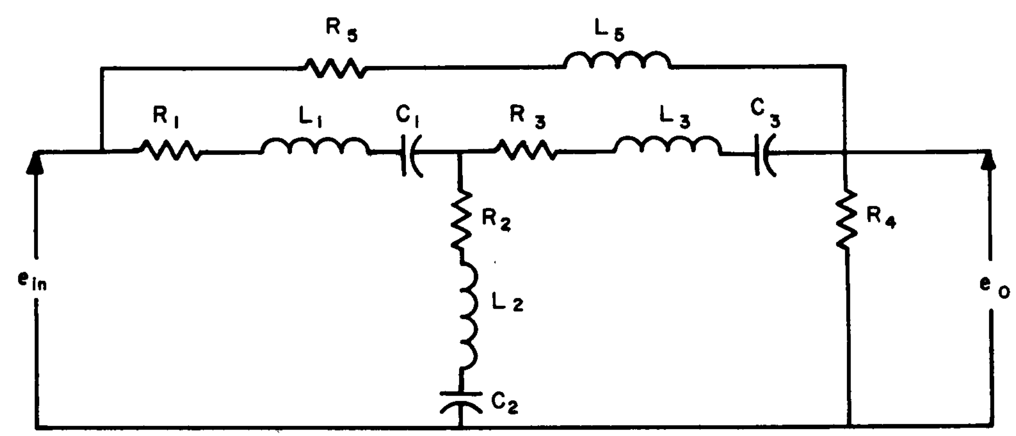
$$\begin{aligned}
D_3 = \; & R_1R_3R_4 + R_4L_1/C_3 + R_4L_3/C_1 + R_3L_4/C_1 + R_1L_4/C_3 + R_3L_1/C_4 + R_1L_3/C_4 + R_1R_2R_4 + R_4L_1/C_2 + \\
& R_4L_2/C_1 + R_2L_4/C_1 + R_1L_4/C_2 + R_2L_1/C_4 + R_1L_2/C_4 + R_2R_4R_5 + R_5L_2/C_4 + R_2L_5/C_4 + R_5L_4/C_2 + \\
& R_4L_5/C_2 + R_2R_3R_4 + R_4L_2/C_3 + R_4L_3/C_2 + R_3L_4/C_2 + R_2L_4/C_3 + R_3L_2/C_4 + R_2L_3/C_4 + R_1R_4R_5 + \\
& R_5L_1/C_4 + R_1L_5/C_4 + R_5L_4/C_1 + R_4L_5/C_1 + R_2R_3R_5 + R_5L_2/C_3 + R_2L_5/C_3 + R_5L_3/C_2 + R_3L_5/C_2 + \\
& R_1R_3R_5 + R_5L_1/C_3 + R_1L_5/C_3 + R_5L_3/C_1 + R_3L_5/C_1 + R_1R_2R_5 + R_5L_1/C_2 + R_1L_5/C_2 + R_5L_2/C_1 + R_2L_5/C_1
\end{aligned}$$

$$\begin{aligned}
D_2 = \; & R_4R_3/C_1 + R_1R_4/C_3 + R_1R_3/C_4 + L_1/C_3C_4 + L_3/C_1C_4 + L_4/C_1C_3 + R_2R_4/C_1 + R_1R_4/C_2 + R_1R_2/C_4 + \\
& L_1/C_2C_4 + L_2/C_1C_4 + L_4/C_1C_2 + R_4R_5/C_2 + R_2R_5/C_4 + L_5/C_2C_4 + R_3R_4/C_2 + R_2R_4/C_3 + R_2R_3/C_4 + \\
& L_2/C_3C_4 + L_3/C_2C_4 + L_4/C_2C_3 + R_4R_5/C_1 + R_1R_5/C_4 + L_5/C_1C_4 + R_3R_5/C_2 + R_2R_5/C_3 + L_5/C_2C_3 + \\
& R_3R_5/C_1 + R_1R_5/C_3 + L_5/C_1C_3 + R_2R_5/C_1 + R_1R_5/C_2 + L_5/C_1C_2
\end{aligned}$$

$$\begin{aligned}
D_1 = \; & R_4/C_1C_3 + R_3/C_1C_4 + R_1/C_3C_4 + R_4/C_1C_2 + R_2/C_1C_4 + R_1/C_2C_4 + R_5/C_2C_4 + R_4/C_2C_3 + R_3/C_2C_4 + \\
& R_2/C_3C_4 + R_5/C_1C_4 + R_5/C_2C_3 + R_5/C_1C_3 + R_5/C_1C_2
\end{aligned}$$

$$D_0 = 1/C_1C_3C_4 + 1/C_1C_2C_4 + 1/C_2C_3C_4$$

Figure IV-15a. Three Loop, 14-Element Circuit



$$T(s) = \frac{N_4S^4 + N_3S^3 + N_2S^2 + N_1S + N_o}{D_5S^5 + D_4S^4 + D_3S^3 + D_2S^2 + D_1S + D_o}$$

where

$$N_4 = R_4L_1L_3 + R_4L_1L_2 + R_4L_2L_5 + R_4L_2L_3$$

$$N_3 = R_4R_3L_1 + R_1R_4L_3 + R_2R_4L_1 + R_1R_4L_2 + R_4R_5L_2 + R_2R_4L_5 + R_3R_4L_2 + R_2R_4L_3$$

$$N_2 = R_1R_3R_4 + R_4L_1/C_3 + R_4L_3/C_1 + R_1R_2R_4 + R_4L_1/C_2 + R_4L_2/C_1 + R_2R_4R_5 + R_4L_5/C_2 + R_2R_3R_4 + R_4L_2/C_3 + R_4L_3/C_2$$

$$N_1 = R_4R_3/C_1 + R_4R_1/C_3 + R_4R_2/C_1 + R_1R_4/C_2 + R_5R_4/C_2 + R_3R_4/C_2 + R_2R_4/C_3$$

$$N_o = R_4/C_1C_3 + R_4/C_1C_2 + R_4/C_2C_3$$

$$D_5 = L_2L_3L_5 + L_1L_3L_5 + L_1L_2L_5$$

Figure IV-15aa. Resistive Load, Three Loop, 12-Element Circuit

Continued:

$$D_4 = R_4L_1L_3 + R_4L_1L_2 + R_4L_2L_5 + R_4L_2L_3 + R_4L_1L_5 + R_5L_2L_3 + R_3L_2L_5 + R_2L_3L_5 + R_5L_1L_3 + R_3L_1L_5 + R_1L_3L_5 + R_5L_1L_2 + R_2L_1L_5 + R_1L_2L_5$$

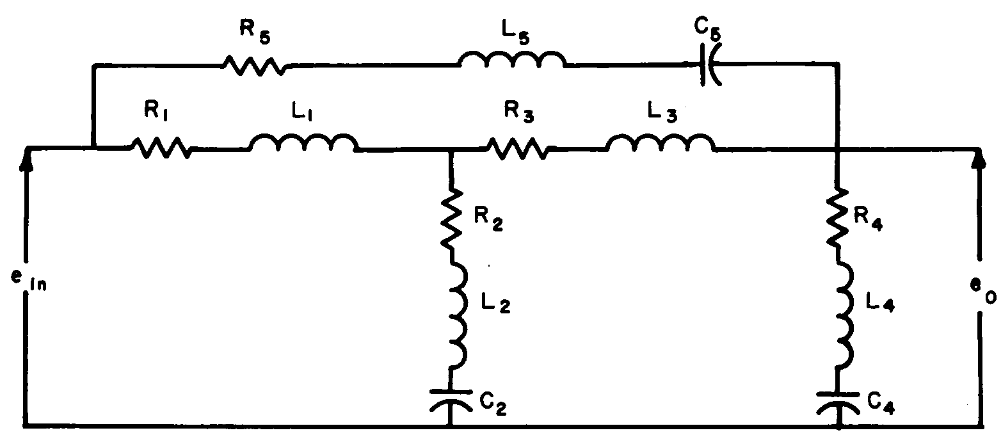
$$D_3 = R_4R_3L_1 + R_4L_1L_3 + R_4R_2L_1 + R_1R_4L_2 + R_4R_5L_2 + R_2R_4L_5 + R_4R_3L_2 + R_2R_4L_3 + R_4R_5L_1 + R_1R_4L_5 + R_3R_5L_2 + R_2R_5L_3 + R_2R_3L_5 + L_2L_5/C_3 + L_3L_5/C_2 + R_5R_3L_1 + R_1R_5L_3 + R_1R_3L_5 + L_1L_5/C_3 + L_3L_5/C_1 + R_2R_5L_1 + R_5R_1L_2 + R_1R_2L_5 + L_1L_5/C_2 + L_2L_5/C_1$$

$$D_2 = R_1R_3R_4 + R_4L_1/C_3 + R_4L_3/C_1 + R_1R_2R_4 + R_4L_1/C_2 + R_4L_2/C_1 + R_2R_4R_5 + R_4L_5/C_2 + R_2R_3R_4 + R_4L_2/C_3 + R_4L_3/C_2 + R_1R_4R_5 + R_4L_5/C_1 + R_2R_3R_5 + R_5L_2/C_3 + R_2L_5/C_3 + R_5L_3/C_2 + R_3L_5/C_2 + R_1R_3R_5 + R_5L_1/C_3 + R_1L_5/C_3 + R_5L_3/C_1 + R_3L_5/C_1 + R_1R_2R_5 + R_5L_1/C_2 + R_1L_5/C_2 + R_5L_2/C_1 + R_2L_5/C_1$$

$$D_1 = R_3R_4/C_1 + R_1R_4/C_3 + R_2R_4/C_1 + R_1R_4/C_2 + R_4R_5/C_2 + R_3R_4/C_2 + R_2R_4/C_3 + R_4R_5/C_1 + R_3R_5/C_1 + R_1R_5/C_3 + L_5/C_1C_3 + R_2R_5/C_1 + R_1R_5/C_2 + L_5/C_1C_2 + R_3R_5/C_2 + R_2R_5/C_3 + L_5/C_2C_3$$

$$D_o = R_4/C_1C_3 + R_4/C_1C_2 + R_4/C_2C_3 + R_5/C_2C_3 + R_5/C_1C_3 + R_5/C_1C_2$$

Figure IV-15aa. Resistive Load, Three Loop, 12-Element Circuit



$$T(s) = \frac{N_6S^6 + N_5S^5 + N_4S^4 + N_3S^3 + N_2S^2 + N_1S + N_o}{D_6S^6 + D_5S^5 + D_4S^4 + D_3S^3 + D_2S^2 + D_1S + D_o}$$

where

$$N_6 = L_1L_3L_4 + L_1L_2L_4 + L_2L_4L_5 + L_2L_3L_4$$

$$N_5 = R_3L_1L_4 + R_1L_3L_4 + R_4L_1L_3 + R_1L_2L_4 + R_4L_1L_2 + R_2L_1L_4 + R_5L_2L_4 + R_4L_2L_5 + R_2L_4L_5 + R_3L_2L_4 + R_4L_2L_3 + R_2L_3L_4$$

$$N_4 = R_1R_3L_4 + R_3R_4L_1 + R_1R_4L_3 + L_1L_3/C_4 + R_1R_4L_2 + R_1R_2L_4 + R_2R_4L_1 + L_1L_2/C_4 + L_1L_4/C_2 + R_4R_5L_2 + R_2R_5L_4 + R_2R_4L_5 + L_2L_5/C_4 + L_4L_5/C_2 + L_2L_4/C_5 + R_3R_4L_2 + R_2R_3L_4 + R_2R_4L_3 + L_2L_3/C_4 + L_3L_4/C_2$$

$$N_3 = R_1R_3R_4 + R_3L_1/C_4 + R_1L_3/C_4 + R_1R_2R_4 + R_1L_2/C_4 + R_2L_1/C_4 + R_1L_4/C_2 + R_4L_1/C_2 + R_2R_4R_5 + R_5L_2/C_4 + R_5L_4/C_2 + R_4L_5/C_2 + R_2L_5/C_4 + R_4L_2/C_5 + R_2L_4/C_5 + R_2R_3R_4 + R_3L_2/C_4 + R_2L_3/C_4 + R_3L_4/C_2 + R_4L_3/C_2$$

Figure IV-15b. Three Loop, 13-Element Circuit

Continued:

$$N_2 = R_1R_3/C_4 + R_1R_4/C_2 + R_1R_2/C_4 + L_1/C_2C_4 + R_4R_5/C_2 + R_2R_5/C_4 + R_2R_4/C_5 + L_2/C_4C_5 + L_4/C_2C_5$$
$$L_5/C_2C_4 + R_3R_4/C_2 + R_2R_3/C_4 + L_3/C_2C_4$$

$$N_1 = R_1/C_2C_4 + R_5/C_2C_4 + R_4/C_2C_5 + R_2/C_4C_5 + R_3/C_2C_4$$

$$N_o = 1/C_2C_4C_5$$

$$D_6 = L_1L_3L_4 + L_1L_2L_4 + L_2L_4L_5 + L_2L_3L_4 + L_1L_4L_5 + L_2L_3L_5 + L_1L_2L_5 + L_1L_3L_5$$

$$D_5 = R_3L_1L_4 + R_1L_3L_4 + R_4L_1L_3 + R_1L_2L_4 + R_4L_1L_2 + R_2L_1L_4 + R_5L_2L_4 + R_4L_2L_5 + R_2L_4L_5 + R_3L_2L_4 +$$
$$R_4L_2L_3 + R_2L_3L_4 + R_3L_1L_5 + R_1L_3L_5 + R_5L_1L_3 + R_1L_4L_5 + R_5L_1L_4 + R_4L_1L_5 + R_3L_2L_5 + R_5L_2L_3 +$$
$$R_2L_3L_5 + R_1L_2L_5 + R_5L_1L_2 + R_2L_1L_5$$

$$D_4 = R_1R_3L_4 + R_3R_4L_1 + R_1R_4L_3 + L_1L_3/C_4 + R_1R_4L_2 + R_1R_2L_4 + R_2R_4L_1 + L_1L_2/C_4 + L_1L_4/C_2 + R_4R_5L_2 +$$
$$R_2R_5L_4 + R_2R_4L_5 + L_2L_5/C_4 + L_4L_5/C_2 + L_2L_4/C_5 + R_3R_4L_2 + R_2R_3L_4 + R_2R_4L_3 + L_2L_3/C_4 +$$
$$L_3L_4/C_2 + R_1R_3L_5 + R_3R_5L_1 + R_1R_5L_3 + L_1L_3/C_5 + R_1R_5L_4 + R_1R_4L_5 + R_4R_5L_1 + L_1L_4/C_5 +$$
$$L_1L_5/C_4 + R_3R_5L_2 + R_2R_3L_5 + R_2R_5L_3 + L_2L_3/C_5 + L_3L_5/C_2 + R_1R_5L_2 + R_1R_2L_5 + R_2R_5L_1 +$$
$$L_1L_2/C_5 + L_1L_5/C_2$$

$$D_3 = R_1R_3R_4 + R_3L_1/C_4 + R_1L_3/C_4 + R_1R_2R_4 + R_1L_2/C_4 + R_2L_1/C_4 + R_1L_4/C_2 + R_4L_1/C_2 + R_2R_4R_5 +$$
$$R_5L_2/C_4 + R_5L_4/C_2 + R_4L_5/C_2 + R_2L_5/C_4 + R_4L_2/C_5 + R_2L_4/C_5 + R_2R_3R_4 + R_3L_2/C_4 + R_2L_3/C_4 +$$
$$R_3L_4/C_2 + R_4L_3/C_2 + R_1R_3R_5 + R_3L_1/C_5 + R_1L_3/C_5 + R_1R_4R_5 + R_1L_4/C_5 + R_4L_1/C_5 + R_1L_5/C_4 +$$
$$R_5L_1/C_4 + R_2R_3R_5 + R_3L_2/C_5 + R_2L_3/C_5 + R_3L_5/C_2 + R_5L_3/C_2 + R_1R_2R_5 + R_1L_2/C_5 + R_2L_1/C_5 +$$
$$R_1L_5/C_2 + R_5L_1/C_2$$

Figure IV-15b. Three Loop, 13-Element Circuit

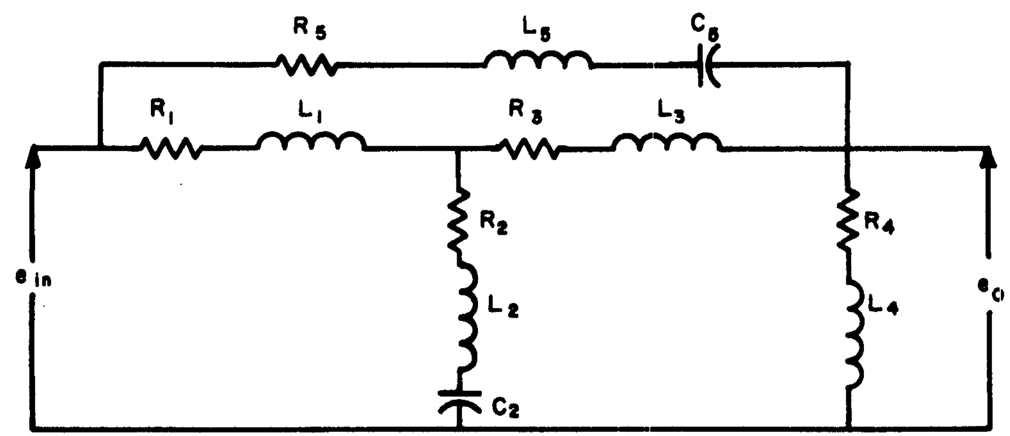
Continued:

$$D_2 = R_3R_1/C_4 + R_1R_4/C_2 + R_1R_2/C_4 + L_1/C_2C_4 + R_4R_5/C_2 + R_2R_5/C_4 + R_2R_4/C_5 + L_2/C_4C_5 + L_4/C_2C_5 +$$
$$L_5/C_2C_4 + R_3R_4/C_2 + R_2R_3/C_4 + L_3/C_2C_4 + R_1R_3/C_5 + R_1R_5/C_4 + R_1R_4/C_5 + L_1/C_4C_5 + R_3R_5/C_2 +$$
$$R_2R_3/C_5 + L_3/C_2C_5 + R_1R_5/C_2 + R_1R_2/C_5 + L_1/C_2C_5$$

$$D_1 = R_1/C_2C_4 + R_5/C_2C_4 + R_4/C_2C_5 + R_2/C_4C_5 + R_3/C_2C_4 + R_1/C_4C_5 + R_3/C_2C_5 + R_1/C_2C_5$$

$$D_o = C_2C_4C_5$$

Figure IV-15b. Three Loop, 13-Element Circuit



$$T(s) = \frac{N_4 S^4 + N_3 S^3 + N_2 S^2 + N_1 S + N_o}{D_5 S^5 + D_4 S^4 + D_3 S^3 + D_2 S^2 + D_1 S + D_o}$$

where

$N_4 = R_4 L_1 L_3 + R_4 L_1 L_2 + R_4 L_2 L_5 + R_4 L_2 L_3$

$N_3 = R_3 R_4 L_1 + R_1 R_4 L_3 + R_1 R_4 L_2 + R_4 R_5 L_2 + R_2 R_4 L_5 + R_3 R_4 L_2 + R_2 R_4 L_2 + R_2 R_4 L_1$

$N_2 = R_1 R_3 R_4 + R_1 R_2 R_4 + R_4 L_1/C_2 + R_2 R_4 R_5 + R_4 L_5/C_2 + R_4 L_2/C_5 + R_2 R_3 R_4 + R_4 L_3/C_2$

$N_1 = R_1 R_4/C_2 + R_4 R_5/C_2 + R_2 R_4/C_5 + R_3 R_4/C_2$

$N_o = R_4/C_2 C_5$

$D_5 = L_2 L_3 L_5 + L_1 L_2 L_5 + L_1 L_3 L_5$

$D_4 = R_4 L_1 L_3 + R_4 L_1 L_2 \; R_4 L_2 L_5 + R_4 L_2 L_3 + R_3 L_1 L_5 + R_1 L_3 L_5 + R_5 L_1 L_3 + R_4 L_1 L_5 + R_3 L_2 L_5 + R_5 L_2 L_3 +$
$R_2 L_3 L_5 + R_1 L_2 L_5 + R_5 L_1 L_2 + R_2 L_1 L_5$

Figure IV-15bb. Resistive Load, Three Loop, 11-Element Circuit

Continued:

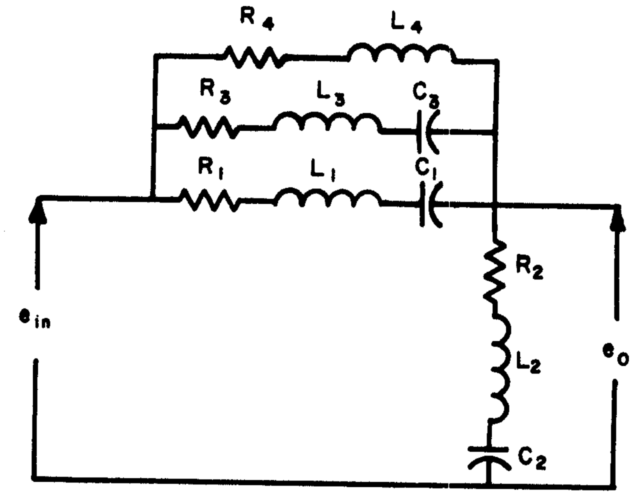
$$D_3 = R_3R_4L_1 + R_1R_4L_3 + R_1R_4L_2 + R_2R_4L_1 + R_2R_4L_5 + R_3R_4L_2 + R_2R_4L_3 + R_1R_3L_5 + R_3R_5L_1 + R_1R_5L_3 +$$
$$L_1L_3/C_5 + R_1R_4L_5 + R_4R_5L_1 + R_3R_5L_2 + R_2R_3L_5 + L_2L_3/C_5 + L_3L_5/C_2 + R_1R_5L_2 + R_1R_2L_5 + R_2R_5L_1 +$$
$$L_1L_2/C_5 + L_1L_5/C_2 + R_4R_5L_2 + R_2R_5L_3$$

$$D_2 = R_1R_3R_4 + R_1R_2R_4 + R_4L_1/C_2 + R_2R_4R_5 + R_4L_5/C_2 + R_4L_2/C_5 + R_2R_3R_4 + R_4L_3/C_2 + R_1R_3R_5 + R_3L_1/C_5$$
$$R_1R_3/C_5 + R_1R_4R_5 + R_4L_1/C_5 + R_2R_3R_5 + R_3L_2/C_5 + R_2L_3/C_5 + R_3L_5/C_2 + R_5L_3/C_2 + R_1R_2R_5 + R_1L_2/C_5$$
$$R_2L_1/C_5 + R_1L_5/C_2 + R_5L_1/C_2$$

$$D_1 = R_1R_4/C_2 + R_4R_5/C_2 + R_2R_4/C_5 + R_3R_4/C_2 + R_1R_3/C_5 + R_1R_4/C_5 + R_3R_5/C_2 + R_2R_3/C_5 + L_3/C_2C_5 +$$
$$R_1R_5/C_2 + R_1R_2/C_5 + L_1/C_2C_3$$

$$D_o = R_4/C_2C_5 + R_3/C_2C_5 + R_1/C_2C_5$$

Figure IV-15bb. Resistive Load, Three Loop, 11-Element Circuit



$$T(s) = \frac{N_6S^6 + N_5S^5 + N_4S^4 + N_3S^3 + N_2S^2 + N_1S + N_o}{D_6S^6 + D_5S^5 + D_4S^4 + D_3S^3 + D_2S^2 + D_1S + D_o}$$

where

$$N_6 = L_1L_2L_3 + L_1L_2L_4 + L_2L_3L_4$$

$$N_5 = R_3L_1L_2 + R_2L_1L_3 + R_1L_2L_3 + R_4L_1L_2 + R_2L_1L_4 + R_1L_2L_4 + R_4L_2L_3 + R_3L_2L_4 + R_2L_3L_4$$

$$N_4 = R_2R_3L_1 + R_1R_3L_2 + R_1R_2L_3 + L_1L_3/C_2 + L_2L_3/C_1 + L_1L_2/C_3 + R_2R_4L_1 + R_1R_4L_2 + R_1R_2L_4 + L_1L_4/C_2 + L_2L_4/C_1 + R_3R_4L_2 + R_2R_4L_3 + R_2R_3L_4 + L_2L_4/C_3 + L_3L_4/C_2$$

$$N_3 = R_1R_2R_3 + R_3L_1/C_2 + R_3L_2/C_1 + R_2L_3/C_1 + R_1L_3/C_2 + R_2L_1/C_3 + R_1L_2/C_3 + R_1R_2R_4 + R_4L_1/C_2 + R_1L_4/C_2 + R_4L_2/C_1 + R_2L_4/C_1 + R_2R_3R_4 + R_4L_2/C_3 + R_2L_4/C_3 + R_4L_3/C_2 + R_3L_4/C_2$$

$$N_2 = R_2R_3/C_1 + R_1R_3/C_2 + R_1R_2/C_3 + L_1/C_2C_3 + L_2/C_1C_3 + L_3/C_1C_2 + R_2R_4/C_1 + R_1R_4/C_2 + L_4/C_1C_2 + R_3R_4/C_2 + R_2R_4/C_3 + L_4/C_2C_3$$

$$N_1 = R_3/C_1C_2 + R_2/C_1C_3 + R_1/C_2C_3 + R_4/C_1C_2 + R_4/C_2C_3$$

Figure IV-16. Three Loop, 11-Element Circuit

Continued:

$$N_o = 1/C_1C_2C_3$$

$$D_6 = L_1L_2L_3 + L_1L_2L_4 + L_2L_3L_4 + L_1L_3L_4$$

$$D_5 = R_3L_1L_2 + R_2L_1L_3 + R_1L_2L_3 + R_4L_1L_2 + R_2L_1L_4 + R_1L_2L_4 + R_4L_2L_3 + R_3L_2L_4 + R_2L_3L_4 + R_4L_1L_3 + R_3L_1L_4 + R_1L_3L_4$$

$$D_4 = R_2R_3L_1 + R_1R_3L_2 + R_1R_2L_3 + L_1L_3/C_2 + L_2L_3/C_1 + L_1L_2/C_3 + R_2R_4L_1 + R_1R_4L_2 + R_1R_2L_4 + L_1L_4/C_2 + L_2L_4/C_1 + R_3R_4L_2 + R_2R_4L_3 + R_2R_3L_4 + L_2L_4/C_3 + L_3L_4/C_2 + R_3R_4L_1 + R_1R_4L_3 + R_1R_3L_4 + L_1L_4/C_3 + L_3L_4/C_1$$

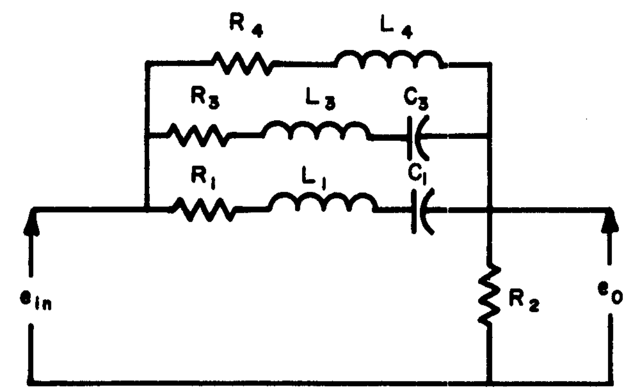
$$D_3 = R_1R_2R_3 + R_3L_1/C_2 + R_3L_2/C_1 + R_2L_3/C_1 + R_1L_3/C_2 + R_2L_1/C_3 + R_1L_2/C_3 + R_1R_2R_4 + R_4L_1/C_2 + R_1L_4/C_2 + R_4L_2/C_1 + R_2L_4/C_1 + R_2R_3R_4 + R_4L_2/C_3 + R_2L_4/C_3 + R_4L_3/C_2 + R_3L_4/C_2 + R_1R_3R_4 + R_4L_1/C_3 + R_1L_4/C_3 + R_4L_3/C_1 + R_3L_4/C_1$$

$$D_3 = R_2R_3/C_1 + R_1R_3/C_2 + R_1R_2/C_3 + L_1/C_2C_3 + L_2/C_1C_3 + L_3/C_1C_2 + R_2R_4/C_1 + R_1R_4/C_2 + L_4/C_1C_2 + R_3R_4/C_2 + R_2R_4/C_3 + L_4/C_2C_3 + R_3R_4/C_1 + R_1R_4/C_3 + L_4/C_1C_3$$

$$D_1 = R_3/C_1C_2 + R_2/C_1C_3 + R_1/C_2C_3 + R_4/C_1C_2 + R_4/C_2C_3 + R_4/C_1C_3$$

$$D_o = 1/C_1C_2C_3$$

Figure IV-16. Three Loop, 11-Element Circuit



$$T(s) = \frac{N_4 S^4 + N_3 S^3 + N_2 S^2 + N_1 S + N_o}{D_5 S^5 + D_4 S^4 + D_3 S^3 + D_2 S^2 + D_1 S + D_o}$$

where

$N_4 = R_2 L_1 L_3 + R_2 L_1 L_4 + R_2 L_3 L_4$

$N_3 = R_2 R_3 L_1 + R_1 R_2 L_3 + R_2 R_4 L_1 + R_2 R_4 L_3 + R_2 R_3 L_4 + R_1 R_2 L_4$

$N_2 = R_1 R_2 R_3 + R_2 L_3/C_1 + R_2 L_1/C_3 + R_1 R_2 R_4 + R_2 L_4/C_1 + R_2 R_3 R_4 + R_2 L_4/C_3$

$N_1 = R_2 R_3/C_1 + R_1 R_2/C_3 + R_2 R_4/C_1 + R_2 R_4/C_3$

$N_o = R_2/C_1 C_3$

$D_5 = L_1 L_3 L_4$

$D_4 = R_2 L_1 L_3 + R_2 L_1 L_4 + R_2 L_3 L_4 + R_4 L_1 L_3 + R_3 L_1 L_4 + R_1 L_3 L_4$

$D_3 = R_2 R_3 L_1 + R_1 R_2 L_3 + R_2 R_4 L_1 + R_1 R_2 L_4 + R_2 R_4 L_3 + R_2 R_3 L_4 + R_3 R_4 L_1 + R_1 R_4 L_3 + R_1 R_3 L_4 + L_1 L_4/C_3 +$
$L_3 L_4/C_1$

Figure IV-17. Resistive Load, Three Loop, 9-Element Circuit

Continued:

$$D_2 = R_1R_2R_3 + R_2L_3/C_1 + R_2L_1/C_3 + R_1R_2R_4 + R_2L_4/C_1 + R_2R_3R_4 + R_2L_4/C_3 + R_1R_3R_4 + R_4L_1/C_3$$
$$R_1L_4/C_3 + R_4L_3/C_1 + R_3L_4/C_1$$

$$D_1 = R_2R_3/C_1 + R_1R_2/C_3 + R_2R_4/C_1 + R_2R_4/C_3 + R_3R_4/C_1 + R_1R_4/C_3 + L_4/C_1C_3$$

$$D_o = R_2/C_1C_3 + R_4/C_1C_3$$

Figure IV-17. Resistive Load, Three Loop, 9-Element Circuit

SECTION V

MINIMIZATION TECHNIQUES[15]

The problem of minimizing the error function E of the several variables $x_1, x_2, \ldots x_n$, which possesses continuous first and second partial derivatives with respect to these variables, was restricted in this effort to the consideration of three basic techniques:[16] 1) the tangent descent method; 2) the Southwell relaxation technique; and 3) the method of steepest descent, or "method of gradients." Since manual computation was used exclusively to determine the element values, the first two techniques proved to be most useful, since they involved only the first partial derivatives. However, with the digital computer, the method of steepest descent (requiring mixed second partials) would be used, since practically no computational discrimination on the part of the computer is required.

The general problem of determining the actual element values will be considered from a minimization point of view. The following items are discussed in this section.

- Inequality Constraints
- Quadratic Program

A. INEQUALITY CONSTRAINTS

The algorithm for performing the search for the minimum of the error function E, with respect to all the variables x_i, reduces in general to: 1) carefully specifying the initial or starting point x_{i_0} about which one is interested, and 2) searching about this point until the error criterion is satisfied. As long as the resulting x_i's represents physically realizable elements, the technique for obtaining the extremum point is an extension of the classical techniques frequently employed to compute the unconstrained minimum of a function of many variables. The consideration that all variables must be bounded leads immediately to the problem of minimization with inequality constraints.

In this effort, the constraint inequality limits the variables to only positive values. The minimum is therefore a point in the interior of the constraint set. The allowable direction of motion towards the minimum point is along any surfaces or constraint surfaces which contain the x_i's.

Figure V-1 illustrates the case of two variables with inequality constraints. The variable x_1 is a bounded variable having a least upper bound U_1 and a greatest lower bound L_1 such that $L_1 \le x_1 \le U_1$. The variable x_2 possesses a least upper bound U_2 and an inequality constraint on the lower end such that $0 < x_2 \le U_2$. Because of the non-linear aspects of the problem, the relative minimum of the objective error function as determined by paths contained on the surfaces must be tested to determine that no other point gives a lower value of the objective function. At present the test consists of simply defining a new set of initial values at a point near the original initial point and minimizing to determine if the process converges to the same local extremum point.

B. QUADRATIC PROGRAM

To achieve the minimization of Φ, a variation of the "gradient technique" called "descent along a tangent" will be applied to assure convergence to a minimum.* It involves a systematic procedure of arriving at Φ_{min} through successive evaluations of Φ at each change in the unknown parameters starting with an initial guess of the variables. The iteration is continued until Φ begins to increase, and then the iteration method is changed to the "Taylor Series Approach" until the minimum is found.[17]

In the descent along a tangent method, Φ is approximated by a linear function

$$\Phi = \Phi_o + \epsilon_n \Phi_n \tag{V-1}$$

where: ϵ_n represents a change in the variable x_i in the direction of the normal to Φ = constant

---

* See Appendix "A" for geometric illustration of the derivation of equations.

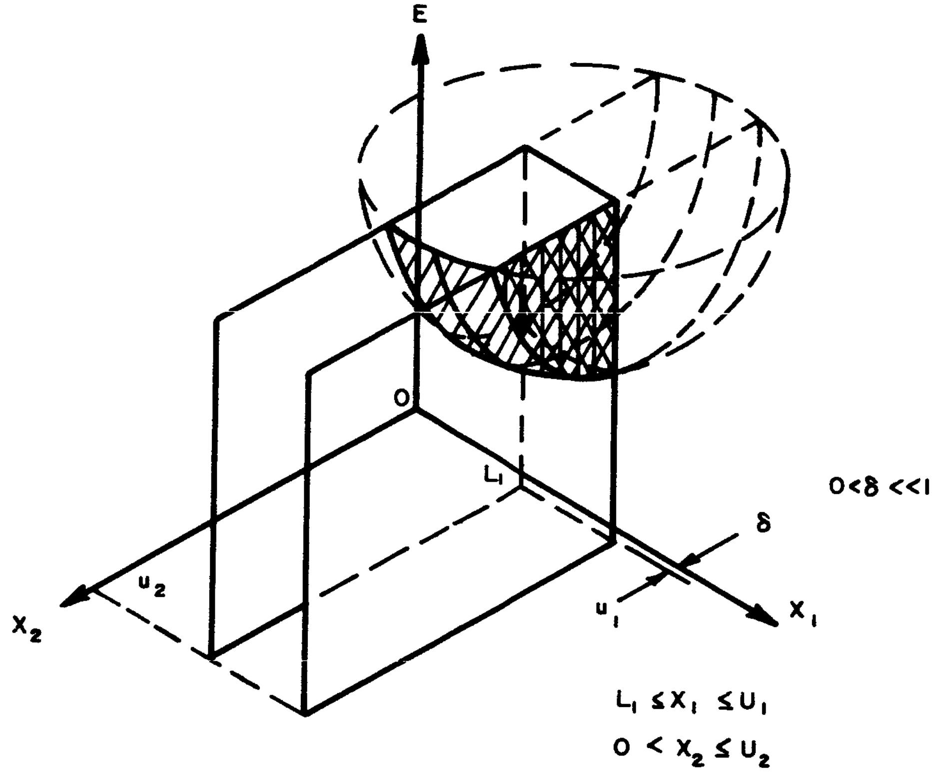


Figure V-1. Surfaces for Minimization with Inequality Constraint

and:

$$\Phi_n = \frac{\partial \Phi}{\partial n} = |\text{ grad } \Phi| = \left\{ \sum_{i=0}^{n} \left(\frac{\partial \Phi}{\partial x_i} \right)^2 \right\}^{\frac{1}{2}} \tag{V-2}$$

For $\Phi = 0$, we have:

$$\epsilon_n = - \frac{\Phi_o}{\Phi_n} \tag{V-3}$$

The changes in the individual coordinates x_i, are given by:

$$\epsilon_i = \epsilon_n \cos (n, x_i) \tag{V-4}$$

where: (n, x_i) is the angle between the normal n, and the x_i axis.

so that:

$$\cos(n, x_i) = \frac{\Phi_i}{\Phi_n}, \text{ and } \Phi_i = \frac{\partial \Phi_o}{\partial x_i} \tag{V-5}$$

Combining Equations (V-2) - (V-5), we obtain:

$$\epsilon_i = -\frac{\Phi_o \; \Phi_i}{\left[\sum_{i=1}^{n} (\Phi_i)^2\right]} \tag{V-6}$$

Change all coordinates x_i to $x_i + \epsilon_i$ and solve for Φ. Repeat the process until Φ is minimized, or until a point is reached at which Φ increases; then use the Taylor Series Approach, repeating the whole process until there is no significant change with minimum of Φ.

In the Taylor Series Approach, the least squares optimization is the minimization of the sum of the squares of the difference between the desired function and its linear approximation at the sampled points.

Expanding Equation (III-17) in a Taylor Series consisting of the linear terms in ϵ_i, i.e., all terms of order higher than the first being omitted:

$$T(j\omega_r) \approx T_o(j\omega_r) + \sum_{i=1}^{n} \frac{\partial T_o(j\omega_r)}{\partial x_i} \epsilon_i \approx T_o(j\omega_r) + \sum_{i=1}^{n} T_i(j\omega_r)\,\epsilon_i \tag{V-7}$$

where the subscript in T_i denotes partial differentiation and is the i^{th} derivative evaluated at the r^{th} sampled point. Rewriting Equation (V-7) by transposing all terms on the right:

$$T(j\omega_r) - \left[T_o(j\omega_r) + \sum_{i=1}^{n} T_i(j\omega_r)\,\epsilon_i\right] \approx 0 \tag{V-8}$$

Combining Equations (V-8) and (III-20), we obtain:

$$\epsilon(\omega_r) - \sum_{i=1}^{n} T_i(j\omega_r)\,\epsilon_i \approx 0 \quad (\text{for all } r = 1, 2, \ldots, n) \tag{V-9}$$

which can be written in expanded form as:

$$\sum_{r=1}^{m} \left\{\left[R(\omega_r) - R_o(\omega_r) - \sum_{i=1}^{n} \frac{\partial R_o(\omega_r)}{\partial x_i} \epsilon_i\right] + j\left[I(\omega_r) - I_o(\omega_r) - \sum_{i=1}^{n} \frac{\partial I_o(\omega_r)}{\partial x_i} \epsilon_i\right]\right\} = 0 \tag{V-10}$$

so that the quantity to be minimized in the least squares sense, redefining the symbolism used in Equation (III-23), is:

$$M = \sum_{r=1}^{m} \left\{ \left[R(\omega_r) - R_o(\omega_r) - \sum_{i=1}^{n} \frac{\partial R_o(\omega_r)}{\partial x_i} \epsilon_i \right]^2 + \left[I(\omega_r) - I_o(\omega_r) - \sum_{i=1}^{n} \frac{\partial I_o(\omega_r)}{\partial x_i} \epsilon_i \right]^2 \right\} \qquad \text{(V-11)}$$

Now, for M minimum, all its partial derivatives with respect to the variable ϵ_i must be equated to zero. This yields:

$$\frac{\partial M}{\partial \epsilon_i} = \sum_{r=1}^{m} \left\{ \left[R(\omega_r) - R_o(\omega_r) - \sum_{i=1}^{n} \frac{\partial R_o(\omega_r)}{\partial x_i} \epsilon_i \right] + \left[\frac{\partial R_o(\omega_r)}{\partial x_i} \right] + \left[(\omega_r) - I_o(\omega_r) - \sum_{i=1}^{n} \frac{\partial I_o(\omega_r)}{\partial x_i} \epsilon_i \right] \left[\frac{\partial I_o(\omega_r)}{\partial x_i} \right] \right\} = 0 \qquad \text{(V-12)}$$

a set of n (linear in ϵ_i) simultaneous equations which must be solved for all i. The steps for the solution of Equation (V-12) are readily programmed for solution by the digital computer. The advantage of the Taylor Series Approach is that it does not require the computation of the second partial derivatives.

In the n dimensional case, the simultaneous Equations (V-12) become:

$$\sum_{r=1}^{m} \left\{ \left[R(\omega_r) - R_o(\omega_r) \right] \left[\frac{\partial R_o(\omega_r)}{\partial x_i} \right] + \left[I(\omega_r) - I_o(\omega_r) \right] \left[\frac{\partial I_o(\omega_r)}{\partial x_i} \right] \right.$$

$$\left. = \sum_{r=1}^{m} \left\{ \frac{\partial R_o(\omega_r)}{\partial x_i} \left[\sum_{i=1}^{n} \frac{\partial R_o(\omega_r)}{\partial x_i} \epsilon_i \right] + \frac{\partial I_o(\omega_r)}{\partial x_i} \left[\sum_{i=1}^{n} \frac{\partial I_o(\omega_r)}{\partial x_i} \epsilon_i \right] \right\} \right\} \qquad \text{(V-13)}$$

evaluated for all i.

The entire procedure for the computational steps required to obtain a solution can best be shown when applied to a specific example. Consider the Rate Gyro Filter (Figures II-3 and II-4), having previously determined the transfer function to be:

$$T(s) = \frac{1}{\left[1 + 0.2\left(\frac{s}{3.5}\right) + \left(\frac{s}{3.5}\right)^2 \right]} = \frac{1}{0.0625\, s^2 + 0.0125 s + 1} \qquad \text{(V-14)}$$

Note that:

$$M_1 = 1, M_2 = (0.0625\, s^2 + 1),\ N_1 = 0,\ N_2 = 0.0125\, s \tag{V-15}$$

So that, from Equation (III-16) we get:

$$\text{Ev } T(s) = \frac{M_1 M_2 - N_1 N_2}{M_2^2 - N_2^2} = \frac{\left[\ 0.0625\, s^2 + 1\ \right]}{\left[0.0625\, s^2 + 1\right]^2 - \left[0.0125\, s\right]^2} \tag{V-16}$$

$$\text{Od } T(s) = \frac{N_1 M_2 - N_2 M_1}{M_2^2 - N_2^2} = \frac{\left[\ -0.0125\, s\ \right]}{\left[0.0625\, s^2 + 1\right]^2 - \left[0.0125\, s\right]^2} \tag{V-17}$$

Along the $j\omega$ axis, Equation (V-16) is strictly a real function of ω and Equation (V-17) is an imaginary function of ω. Therefore, reinterpreting Equations (V-16) and (V-17) in light of Equation (III-17), we obtain:

$$R(\omega_r) = \frac{\left[1 - 0.0625\, \omega_r^2\right]}{\left[1 - 0.0625\, \omega_r^2\right]^2 + \left[0.0125\, \omega_r\right]^2} \tag{V-18}$$

$$I(\omega_r) = \frac{-\left[0.0125\, \omega_r\right]}{\left[1 - 0.0625\, \omega_r^2\right]^2 + \left[0.0125\, \omega_r\right]^2} \tag{V-19}$$

Next, determine the network resulting in required order of transfer function. One such network meeting the requirements with specified termination is:

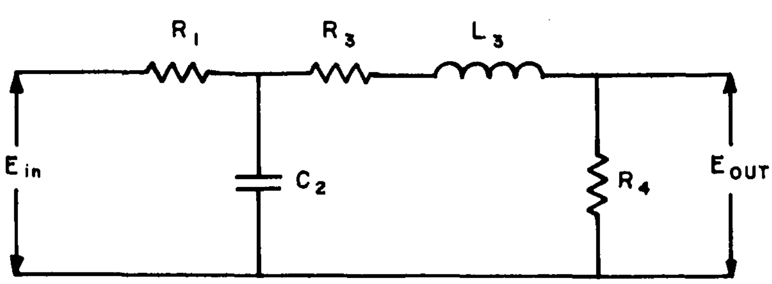


Figure V-2. Terminated Network

Writing Equation (III-18) in terms of the above circuit gives the transfer function:

$$T_o(s) = \frac{E_{out}}{E_{in}}(s) = \frac{R_4}{(R_1 L_3 C_2)\, s^2 + (L_3 + R_1 R_4 C_2 + R_1 R_3 C_2)\, s + (R_4 + R_3 + R_1)} \tag{V-20}$$

Making the substitution of x_i in Equation (V-20) for the dependent variables results in:

$$x_1 = R_1$$
$$x_2 = C_2$$
$$x_3 = R_3$$
$$x_4 = L_3$$
$$x_5 = R_4$$

$$T_o(s) = \frac{x_5}{(x_1x_2x_4)\,s^2 + (x_4 + x_1x_2x_5 + x_1x_2x_3)s + (x_1 + x_3 + x_5)} \qquad \text{(V-21)}$$

Solving Equation (V-21) for real and imaginary terms as required by Equation (III-19), we get:

$$R_o(\omega_r) = \frac{x_5\left[x_1 + x_3 + x_5 - (x_1x_2x_4)\,\omega_r^2\right]}{\left[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^2\right]^2 + \left[(x_4+x_1x_2x_5 + x_1x_2x_3)\,\omega_r\right]^2} \qquad \text{(V-22)}$$

$$I_o(\omega_r) = \frac{-x_5\left[(x_4 + x_1x_2x_5 + x_1x_2x_3)\,\omega_r\right]}{\left[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^2\right]^2 + \left[(x_4+x_1x_2x_5 + x_1x_2x_3)\,\omega_r\right]^2} \qquad \text{(V-23)}$$

We now let Φ_o be the value of Φ in Equation (III-23) at the first approximation of the variables x_1, x_2, x_3, x_4, and where x_5 is fixed at the value of the required resistive load.

Then:

$$\Phi_o = \sum_{r=1}^{m}\left\{\left[R(\omega_r) - R_o(\omega_r)\right]^2 + \left[I(\omega_r) - I_o(\omega_r)\right]^2\right\} \qquad \text{(V-24)}$$

In the method of steepest descent along a tangent, we calculate increments from Equation (V-6), here repeated:

$$\epsilon_i = -\frac{\Phi_o\,\Phi_i}{\left[\sum_{i=1}^{n}(\Phi_i)^2\right]} \qquad \text{(V-6)}$$

where Φ_o is the value of Φ at the initial approximation.

Also in Equation (V-6):

$$\Phi_i = \frac{\partial \Phi}{\partial x_i} \tag{V-25}$$

the value found from the initial approximation and denoted by: Φ_{i_o} .

Therefore, evaluating Equation (V-25) we get:

$$\begin{aligned} \Phi_{i_o} = \frac{\partial \Phi_o}{\partial x_i} = \sum_{r=1}^{m} \Bigg\{ 2 \Big[R(\omega_r) - R_o(\omega_r)\Big]\Big[-\frac{\partial R_o(\omega_r)}{\partial x_i}\Big] \\ + 2\Big[-I(\omega_r) - I_o(\omega_r)\Big]\Big[-\frac{\partial I_o(\omega_r)}{\partial x_i}\Big]\Bigg\} \end{aligned} \tag{V-26}$$

for all i = 1, 2, ..., n.

In Equation (V-26), the first partial with respect to x_i of Equation (V-22) is:

$$\frac{\partial R_o(\omega_r)}{\partial x_i} = \frac{\begin{gathered} \Big\{\Big[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^{\,2}\Big]^2 + \Big[(x_4+x_1x_2x_5 + x_1x_2x_3)\,\omega_r\Big]^2\Big\} \\ \Big\{x_5 \frac{\partial}{\partial x_1}\Big[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^{\,2}\Big]\Big\} - \Big\{x_5\Big[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^{\,2}\Big]\Big\} \\ \Big\{2\Big[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^{\,2}\Big]\Big(\frac{\partial}{\partial x_i}\Big[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^{\,2}\Big]\Big) \\ + 2\Big[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r\Big]\Big(\frac{\partial}{\partial x_1}\Big[(x_4+x_1x_2x_5 + x_1x_2x_3)\,\omega_r\Big]\Big)\Big\} \end{gathered}}{\Big\{\Big[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^{\,2}\Big]^2 + \Big[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r^{\,2}\Big]^2\Big\}} \tag{V-27}$$

Also in Equation (V-26), the first partial with respect to x_i of Equation (V-23) is:

$$\frac{\partial I_o(\omega_r)}{\partial x_i} = \frac{\begin{gathered} \Big\{\Big[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^{\,2}\Big]^2 + \Big[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r\Big]^2\Big\} \\ \Big\{-x_5 \frac{\partial}{\partial x_i}\Big[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r\Big]\Big\} - \Big\{-x_5\Big[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r\Big]\Big\} \\ \Big\{2\Big[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^{\,2}\Big]\Big(\frac{\partial}{\partial x_i}\Big[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^{\,2}\Big]\Big) \\ + 2\Big[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r\Big]\Big(\frac{\partial}{\partial x_i}\Big[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r\Big]\Big)\Big\} \end{gathered}}{\Big\{\Big[x_1+x_3+x_5 - (x_1x_2x_4)\,\omega_r^{\,2}\Big]^2 + \Big[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r^{\,2}\Big]^2\Big\}} \tag{V-28}$$

and for the solution of Equations (V-27) and (V-28), we required the indicated partial derivations of the variable x_i for all i:

$$
\text{a.}\left\{\begin{array}{l}
\frac{\partial}{\partial x_1}\left\{\left[x_1+x_3+x_5-(x_1x_2x_4)\,\omega_r^2\right]\right\}=1-(x_2x_4)\,\omega_r^2 \\
\frac{\partial}{\partial x_2}\left\{\left[x_1+x_3+x_5-(x_1x_2x_4)\,\omega_r^2\right]\right\}=-(x_1x_4)\,\omega_r^2 \\
\frac{\partial}{\partial x_3}\left\{\left[x_1+x_3+x_5-(x_1x_2x_4)\,\omega_r^2\right]\right\}=1 \\
\frac{\partial}{\partial x_4}\left\{\left[x_1+x_3+x_5-(x_1x_2x_4)\,\omega_r^2\right]\right\}=-(x_1x_2)\,\omega_r^2
\end{array}\right. \tag{V-29}
$$

$$
\text{b.}\left\{\begin{array}{l}
\frac{\partial}{\partial x_1}\left\{\left[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r\right]\right\}=(x_2x_5+x_2x_3)\,\omega_r \\
\frac{\partial}{\partial x_2}\left\{\left[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r\right]\right\}=(x_1x_5+x_1x_3)\,\omega_r \\
\frac{\partial}{\partial x_3}\left\{\left[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r\right]\right\}=(x_1x_2)\,\omega_r \\
\frac{\partial}{\partial x_4}\left\{\left[(x_4+x_1x_2x_5+x_1x_2x_3)\,\omega_r\right]\right\}=\omega_r
\end{array}\right.
$$

For the solution of Equation (V-6), Equation (V-26) has to be determined.

Substitution of Equations (V-27), (V-28), (V-18), (V-19), (V-22), and (V-23) into (V-26), we obtain its solution for use in Equation (V-6). With Equation (V-6) solved, we now add these increments to the values used in our initial approximation, changing notation so that zero subscripts refer to the new approximations, and recomputing the solution to Equation (V-24)

$$x_{i_o} = x_i + \epsilon_i \tag{V-30}$$

We iterate with this method until Equation (V-24) increases. When this occurs, we resort to the Taylor Series Approach defined by Equation (V-13) with those values of the variables as initial approximation used in the iteration step preceding the increase in Equation (V-24) of the tangent descent process.

Equation (V-13) can be written in matrix form as:

$$(A_R^T A_R + A_I^T A_I)\ \epsilon = A_R^T E_R + A_I^T E_I \tag{V-31}$$

where:

$$A_R = \begin{bmatrix} R_1(\omega_1) & R_2(\omega_1) \ldots\ldots & R_n(\omega_1) \\ R_1(\omega_2) & R_2(\omega_2) \ldots\ldots & R_n(\omega_2) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ R_1(\omega_m) & R_2(\omega_m) \ldots\ldots & R_n(\omega_m) \end{bmatrix}$$

$$A_I = \begin{bmatrix} I_1(\omega_1) & I_2(\omega_1) \ldots\ldots & I_n(\omega_1) \\ I_1(\omega_2) & I_2(\omega_2) \ldots\ldots & I_n(\omega_2) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ I_1(\omega_m) & I_2(\omega_m) \ldots\ldots & I_n(\omega_m) \end{bmatrix}$$

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_n \end{bmatrix}$$

$$E_R = \begin{bmatrix} R_D(\omega_1) \\ R_D(\omega_2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ R_D(\omega_m) \end{bmatrix}$$

$$E_I = \begin{bmatrix} I_D(\omega_1) \\ I_D(\omega_2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ I_D(\omega_m) \end{bmatrix}$$

and:

$R_D(\omega_r) = R(\omega_r) - R_o(\omega_r)$	is the real part of the error evaluated at the r^{th} sampled point.
$I_D(\omega_r) = I(\omega_r) - I_o(\omega_r)$	is the imaginary part of the error evaluated at the r^{th} sampled point.
$R_i(\omega_r) = \dfrac{\partial R_o(\omega_r)}{\partial x_i}$	is the i^{th} derivative of the imaginary part of $T_o(j\omega_r)$ evaluated at the r^{th} sampled point.
$I_i(\omega_r) = \dfrac{\partial I_o(\omega_r)}{\partial x_i}$	is the i^{th} derivative of the real part of $T_o(j\omega_r)$ evaluated at the r^{th} sampled point.

Equation (V-31) can be solved for the unknown by matrix inversion as:

$$\epsilon = (A_R^T A_R + A_I^T A_I)^{-1} (A_R^T E_R + A_I^T E_I) \tag{V-32}$$

The required mathematical steps for solution of Equation (V-32) are:

(1) Compute the derivatives $R_i(\omega_r)$ and $I_i(\omega_r)$.

(2) Solve for the errors E_R and E_I.

(3) Perform indicated matrix multiplication, addition, and matrix inversion.

The above result in the solution of the set of n simultaneous equations which yield the new approximation $(x_i \rightarrow x_i + \epsilon_i)$ to the minimum.

SECTION VI

EXAMPLE

This section is expository in nature, since a quadratic lag function synthesis is considered in detail and a comparison is made between the various methods used to develop the circuit and the resulting elements. A circuit for a simple quadratic lag function $\left[\dfrac{1}{Q\left(\frac{0.25}{4}\right)}\right]$ requirement as shown in Figures II-5 and II-6 is synthesized, first by the "classical method" and then by the techniques presented in this work. The following matters will be discussed in this section:

- Classical Technique
- First Synthesis
- Second Synthesis
- Comparison

A. CLASSICAL TECHNIQUE

This synthesis technique considers the make-up of the network as a tandem-connected sequence of constant resistance sections, each one imposing constraints on the over-all transfer function. It will be shown in the procedure that to be physically realizable, the simple quadratic lag function must be modified by surplus factors, resulting in a more complex form for the transfer function.

1. Physical Realizability Conditions

It can be shown that for a transfer function to be realizable by passive elements in the form of tandem-connected stages in a constant resistance ladder network, it must possess the following properties:

- Be minimum phase.
- Have no poles on the imaginary axis.
- Be positive real (pr).

It may be noted by an inspection of Equations (VI-1) and (VI-2) that the first two conditions are readily satisfied.

$$\frac{1}{Q\left(\frac{0.25}{4}\right)} = \frac{1}{0.0625\ S^2 + 0.125\ S + 1} \tag{VI-1}$$

$$= \frac{1}{\left[S + (1 + j\sqrt{15})\right]\left[S + (1 - j\ \sqrt{15})\right]} \tag{VI-2}$$

A constant gain factor, K, is introduced in the transfer function which must be determined to make the branch impedances realizable. (It is accounted for by introducing a compensating gain factor for the over-all ladder network.) To test for pr, consider the transfer function in the form:

$$T(s) = K\left[\frac{1}{K(0.0625\ s^2 + 0.125\ s + 1)}\right] = K\left[t\ (s)\right] \tag{VI-3}$$

where $$t\ (s) = \left[\frac{1}{K\ (0.0625\ s^2 + 0.125\ s + 1)}\right] \tag{VI-4}$$

substituting $s = j\ \omega$ results in

$$t\ (j\,\omega) = \frac{16}{K}\left[\frac{(16 - \omega^2) - j\ 2\ \omega}{(16 - \omega^2)^2 + (2\omega)^2}\right] = R\ (\omega) + j\ I\ (\omega) \tag{VI-5}$$

therefore, the real part is:

$$R\ (\omega) = \frac{16}{K}\left[\frac{(16 - \omega^2)}{(16 - \omega^2)^2 + (2\omega)^2}\right] = \frac{N(\omega^2)}{D(\omega^2)} \tag{VI-6}$$

Now, noting that for $\omega \geq 0$ the denominator is positive and the constants in front of the brackets are also positive, one has to determine the conditions, if any, on the numerator for all $\omega \geq 0$, to determine positive realness.

$$N\ (\omega^2) = 16 - \omega^2 \tag{VI-7}$$

now, for $N\ (\omega^2)$ to be pr

$$16 \geq \omega^2 \text{ for all } \omega \geq 0 \tag{VI-8}$$

therefore, since the above condition is not satisfied for all possible values of ω, the given transfer function is not pr.

Multiplying the numerator and denominator of the above function by a factor $(S + \alpha)$, a pr transfer function may be determined, leading to a physically realizable network configuration consisting of passive elements.

Therefore:

$$t(s) = \frac{16}{K}\left[\frac{1}{s^2 + 2s + 16}\right] \tag{VI-9}$$

$$= \left[\frac{(s+\alpha)}{K_1(s^2 + 2s + 16)}\right]\left[\frac{1}{K_2\,(s+\alpha)}\right] = t_1(s) \cdot t_2(s) \tag{VI-10}$$

where the particular value of α will be chosen so as to simplify the computation and still satisfy the pr conditions, and:

$$\frac{16}{K} = \frac{1}{K_1} \cdot \frac{1}{K_2} \tag{VI-11}$$

It may now be noted that the fundamental transfer function has been made more complex to meet the restrictions imposed by the physical realizability conditions. At this point, the constant resistance ladder network consists of at least two cascade stages, defined by the transfer functions $t_1(s)$ and $t_2(s)$.

Determining the conditions for $t_1(s)$ to be pr, by substituting $s = j\,\omega$ results in:

$$t_1(j\,\omega) = \frac{\left[16\,\alpha + \omega^2\,(2 - \alpha)\right] + j\,\omega\left[16 - 2\alpha - \omega^2\right]}{K_1\left[(16 - \omega^2)^2 + 4\,\omega^2\right]} = R(\omega) + jI(\omega) \tag{VI-12}$$

therefore, the real part is:

$$R(\omega) = \frac{\left[16\,\alpha + \omega^2\,(2 - \alpha)\right]}{K_1\left[(16 - \omega^2)^2 + 4\omega^2\right]} = \frac{N_1\,(\omega)}{D_1\,(\omega)} \tag{VI-13}$$

Since K is a positive definite and the denominator is positive for all $\omega \geq 0$ the transfer function will be pr for the condition:

$$\alpha \leq 2 \tag{VI-14}$$

Therefore for ease of computation assume an:

$$\alpha = 1 \tag{VI-15}$$

resulting in a transfer function of the form:

$$t\ (s) = \left[\frac{s + 1}{K_1\ (s^2 + 2s + 16)}\right]\left[\frac{1}{K_2\ (s + 1)}\right] = t_1\ (s) \cdot t_2\ (s) \qquad \text{(VI-16)}$$

To determine the value for K_1, it may be shown that for a lead quadratic of the general form $\dfrac{s + \alpha}{K\ (s^2 + 2\zeta\omega_n s + \omega_n^2)}$, the value for K may be determined by taking the largest value determined by the following equation:

$$K = \text{maximum}\left(\frac{\alpha}{\omega_n^2}\ ,\ \frac{1}{2\zeta\ \omega_n - \alpha}\right) \qquad \text{(VI-17)}$$

Therefore, discriminating between the two values calculated from Eq. (VI-17)

$$K_1 = \left\{\begin{matrix} \frac{1}{16} = 0.0625 \\ \\ \frac{1}{2x1\ -\ 1} = 1 \end{matrix}\right\}_{\max} \qquad \text{(VI-18)}$$

results in using:

$$K_1 = 1 \qquad \text{(VI-19)}$$

Since $t_2(s)$ is a pr function, the choice of K_2 was governed by the criterion of simplifying the computation. Therefore, a $K_2 = 1$ was chosen.

2. Determination of Circuit Parameters

The circuit configuration for a cascaded constant resistance ladder network is shown in Figure VI-1. Where:

$$Z_{an} = 1 + \frac{1}{Z_{bn}} \qquad \text{(VI-20)}$$

$$Z_{bn} = \frac{1}{t_n(s)} - 1 \qquad \text{(VI-21)}$$

Using Equations (VI-20) and (VI-21) to calculate Z_{a1} and Z_{b1} results in:

$$Z_{a1} = 1 + \cfrac{}{s + \cfrac{1}{\frac{s}{15} + \frac{1}{15}}} \qquad \text{(VI-22)}$$

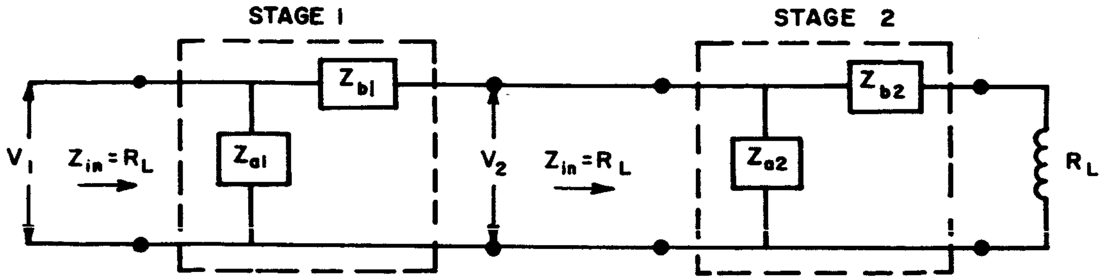


Figure VI-1. Cascaded Constant Resistance Ladder Stages

$$Z_{b1} = \frac{s^2 + 25 + 16}{s + 1} = s + \frac{1}{\frac{s}{15} + \frac{1}{15}} \qquad \text{(VI-23)}$$

which may be interpreted circuitwise as shown in Figure VI-2.

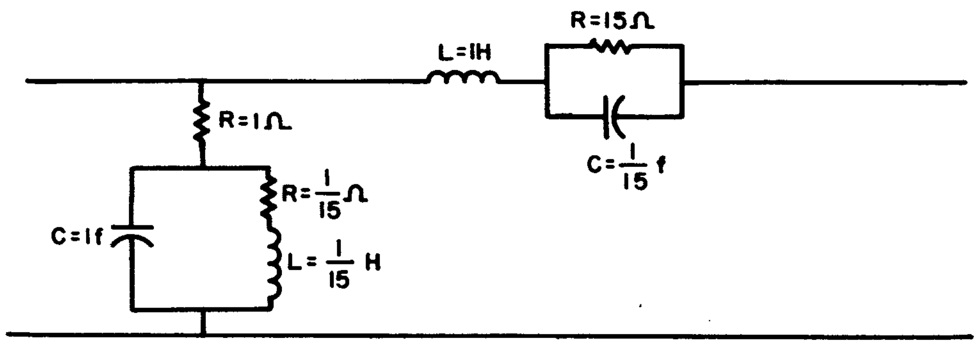


Figure VI-2. First Stage of Ladder Network

In a manner similar to that indicated above, the impedances Z_{a2} and Z_{b2} were calculated as:

$$Z_{a2} = 1 + \frac{1}{s} \qquad \text{(VI-24)}$$

$$Z_{b2} = s \qquad \text{(VI-25)}$$

which may be interpreted as the second stage circuit as shown in Figure VI-3.

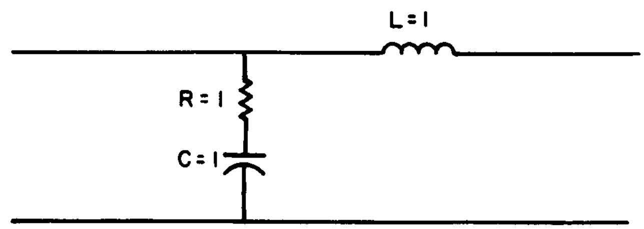


Figure VI-3. Second Stage of Ladder Network

The normalized constant resistance ladder network satisfying the transfer function defined by Equation (VI-1) is indicated in Figure VI-4, with the unnormalized values (R_L = 800 Ω) shown in parentheses. It contains 10 elements, and the over-all gain factor (K) that yields the final required compensation network is given as Equation (VI-16).

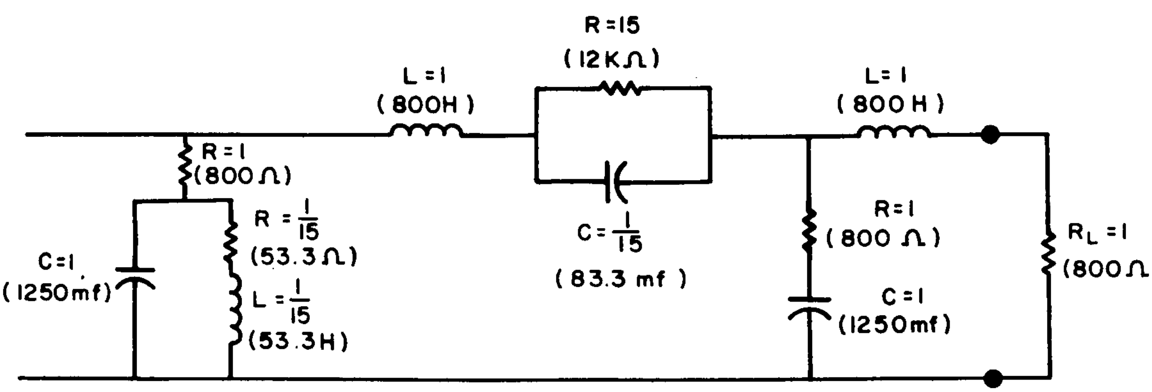


Figure VI-4. Circuit Developed Using Classical Technique

B. FIRST SYNTHESIS

A simple configuration is considered first so as to develop some numerical insight into the problems associated with the synthesis of networks. Figure VI-5 is a diagram of the network, containing its topology and impedances as suggested in Table IV-2.

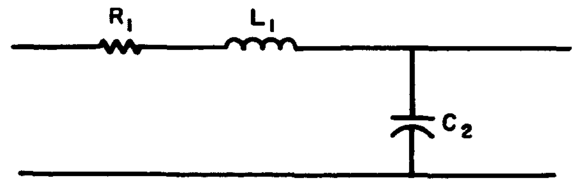


Figure VI-5. One Mesh Circuit Used for $\left[\frac{1}{Q\left(\frac{0.25}{4}\right)}\right]$

Although this configuration is trivial in the sense that the load impedance is not considered, the insight gained from the calculations more than outweighs this criticism.

1. Equating Coefficients

Since the circuit transfer function was of the form:

$$t(s) = \frac{1}{LCs^2 + RCs + 1} \tag{VI-26}$$

it was possible to determine the values for the circuit parameters exactly by the simple expedient of equating coefficients, therefore:

$$\begin{cases} LC = 0.0625 \\ RC = 0.125 \end{cases} \tag{VI-27}$$

Assuming a particular value for the inductance:

$$L = 400 \text{ H} \tag{VI-28}$$

it is now possible to determine the other values uniquely as:

$$R = 800\ \Omega \tag{VI-29}$$

$$C = 156.25 \text{ mf} \tag{VI-30}$$

2. Keeping One Variable Constant

For ease of computation the following normalized values were arbitrarily chosen as the initial point to start the minimization process:

$$P_o(x_1, x_2, x_3) = P_o(R, L, C) = P_o(0.6, 0.4, 0.1) \tag{VI-31}$$

TABLE VI-1

SUMMARY OF FIRST SYNTHESIS ITERATIONS X_2 = CONSTANT

Iteration	$X_1 = R_1$	$X_2 = L_1$	$X_3 = C_2$	Eo
1	.6	.4	.1	.5532536487
2	.6004387384	.4	.1303752154	.1573778386
3	.6008942761	.4	.1473414394	.0386337954
4	.6015637763	.4	.1582284086	.0185647458
$5a_1$	.6645457131	.4	.2046841632	.3492697383
5b	.8006996183	.4	.1582284086	.0006366793
6	.8006996183	.4	.1568967194	.0000697935
$7a_2$	.8006996183	.4	.1565585856	.0000166480
$7b_3$	800	399.65049640H	.156695499mf	-----------

1) Changed from tangent descent to relaxation technique
2) Final normalized values
3) Final denormalized values

The results of the minimization of the error function E are summarized in Table VI-1. A comparison of the final denormalized values, with the exact values determined by equating the coefficients, shows that all values are within tolerable error, and that the error could be decreased by further iterations.

3. Varying All Parameters

As the next step in the development all parameters were varied, starting at the same initial point as previously. After thirty-six iterations only, the final normalized values and final denormalized values are presented in Table VI-2.

TABLE VI-2

SUMMARY OF FIRST SYNTHESIS ITERATIONS VARYING ALL PARAMETERS

	$X_1 = R_1$	$X_2 = L_1$	$X_3 = C_2$	Eo
Normal Value	0.6015241501	0.3176630756	0.1967493373	0.0008948796
Denorm. Value	800 Ω	422.477568H	160.15241501mf	----------

By continuing the iteration process the error, Eo, could be further decreased, bringing the values of the parameters closer to those determined previously.

C. SECOND SYNTHESIS

After considering the trivial single mesh case as noted in Subsection VI-B, a simple two-mesh topology with load impedance was next used as a more realistic approach to the problem. It may be noted that the circuit designer, by using the technique advocated in this work, has at his command a large number of networks, each yielding the required transfer functions. By adhering strictly to the classical approach, the designer has only a limited configuration that meets the rigid requirements which are satisfied in general by ideal elements. Figure VI-6 is a diagram of the simple two-mesh topology and impedances suggested by Table IV-2.

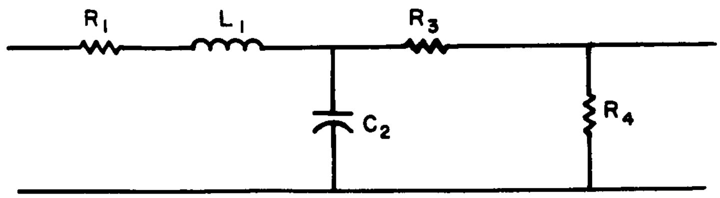


Figure VI-6. Two-Mesh Circuit Used for $\left[\dfrac{1}{Q_2\left(\frac{0.25}{4}\right)}\right]$

The basic idea for setting up the error function E of several variables x_1, x_2, ... x_n is outlined below. The results of the minimization of the error function E are also summarized below (Subsection VI-C-2).

1. "Setting up" E

Step (1) Define incidence matrix: $\left[\ I\ \right] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$

Step (2) Find transpose of incidence matrix: $[I]^T = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

Step (3) Define elements of primitive diagonal matrix in complex form:

$$\left[Z_p\right] = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix} + j \begin{bmatrix} \omega_s L_1 & 0 & 0 & 0 \\ 0 & -\dfrac{1}{\omega_s C_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step (4) Find the characteristic equation by matrix multiplication:

$[\Delta] = [I]\ [Z_p]\ [I]^T$, where:

$$[\Delta] = \left[\Delta_R\ (\omega_s)\right] + j\ \left[\Delta_I\ (\omega_s)\right] = \begin{bmatrix} R_1 & 0 \\ 0 & R_3 + R_4 \end{bmatrix} + j \begin{bmatrix} \omega_s\ L_1 - \dfrac{1}{\omega_s C_2} & \dfrac{1}{\omega_s C_2} \\ \dfrac{1}{\omega_s C_2} & -\dfrac{1}{\omega_s C_2} \end{bmatrix}$$

Step (5) Find minor M_{12} of the determinant of: Δ

$$| M_{12} | = | 0 | + j \left| \frac{1}{\omega_s C_2} \right|$$

Step (6) Using the complex load $Z_L = R_4 + j [0]$ multiply it by the minor $| M_{12} |$ to define the numerator:

$$n_R (\omega_s) + j\, n_I (\omega_s) = 0 + j \frac{R_4}{\omega_s C_2}$$

Step (7) The circuit transfer function t(s) is now defined as:

$$t(s) = \frac{n_R (\omega_s) + j\, n_I (\omega_s)}{\Delta_R (\omega_s) + j\, \Delta_I (\omega_s)}$$

Step (8) Define the error function E as the magnitude function squared which, summed over all discrete values of ω_s, yields:

$$E = \sum_{s=0}^{t} \left\{ R(\omega_s) \left[R_1 (R_3 + R_4) + \frac{L_1}{C_2} \right] - I(\omega_s) \left[\omega_s L_1 (R_3 + R_4) - \frac{1}{\omega_s C_2} (R_1 + R_3 + R_4) \right] \right\}^2$$
$$+ \left\{ R(\omega_s) \left[\omega_s L_1 (R_3 + R_4) - \frac{1}{\omega_s C_2} (R_1 + R_3 + R_4) \right] + I(\omega_s) \left[R_1 (R_3 + R_4) + \frac{L_1}{C_2} \right] - \frac{R_4}{\omega_s C_2} \right\}^2$$

which may be written using the generalized variable x_i as:

$$E = \sum_{s=0}^{t} \left\{ R(\omega_s) \left[x_1 (x_4 + x_5) + \frac{x_2}{x_3} \right] - I(\omega_s) \left[\omega_s x_2 (x_4 + x_5) - \frac{1}{\omega_s x_3} (x_1 + x_4 + x_5) \right] \right\}^2$$
$$+ \left\{ R(\omega_s) \left[\omega_s x_2 (x_4 + x_5) - \frac{1}{\omega_s x_3} (x_1 + x_4 + x_5) \right] + I(\omega_s) \left[x_1 (x_4 + x_5) + \frac{x_2}{x_3} \right] - \frac{x_5}{\omega_s x_3} \right\}^2$$

(VI-32)

2. Evaluated Circuit Parameters

The result of minimizing Equation VI-32 in Step 8 of the preceding yielded the normalized values for the network parameters. Table VI-3 summarizes the results at each iteration of the tangent descent procedure. Step 6b in Table VI-3 contains the denormalized values that refer to the network configuration shown in Figure VI-6.

Since a desk calculator was used throughout this work, every effort was made to decrease the calculations to 10 significant figures. The following approximations were used, since they would not detract from the effectiveness of the technique:

1) The particular frequencies of interest to NASA were located at $\omega = 1.885,\ 7.8540,\ 12.8800,\ 19.1638,\ 23.876$; for ease of computation the frequencies that were used were $\omega = 1,\ 6,\ 12,\ 18,\ 24$.

2) Although the error function should be summed over all the discrete frequencies of interest, another simplification for computational purposes was made in which the frequencies of interest were limited to $\omega = 1$ and $\omega = 6$.

3) By using the values derived from the minimization procedure, the transfer function was of the form:

$$t(s) = 0.864 \left[\frac{1}{Q_2\left(\frac{0.361}{4.47}\right)} \right]$$

This result is considered as an approximation, since the computation was stopped when there was no significant change in the third decimal place of the error function, and no further refined calculations (such as the gradient technique) were used.

Figures VI-7 and VI-8 compare the required curve (dotted line) with the curve developed using the minimization curve (solid line). It may be noted that for engineering purposes the gain errors are small at the two designated frequencies of interest, and in certain regions there has been an improvement in the phase response.

D. COMPARISON

A direct comparison between Figures VI-4, VI-5, and VI-6 shows that by using classical synthesis techniques, a more complicated network configuration results in the interconnection of ideal (i.e., non-dissipative) branch elements. The simple quadratic lag function that was considered had to be modified by surplus factors (resulting in a more complex form of the transfer function) before

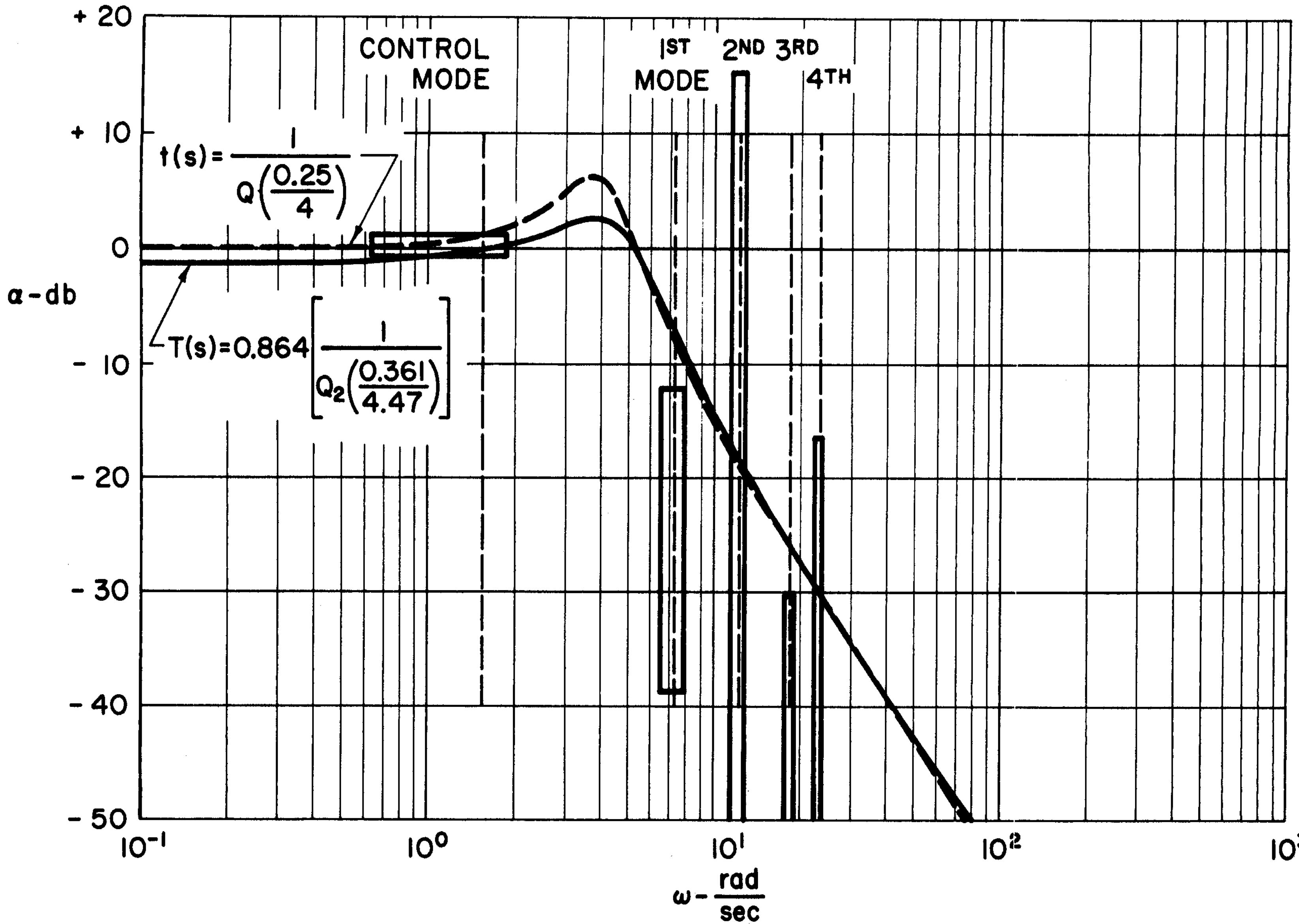


Figure VI-7. Gain Tolerance for Rate Gyro Filter

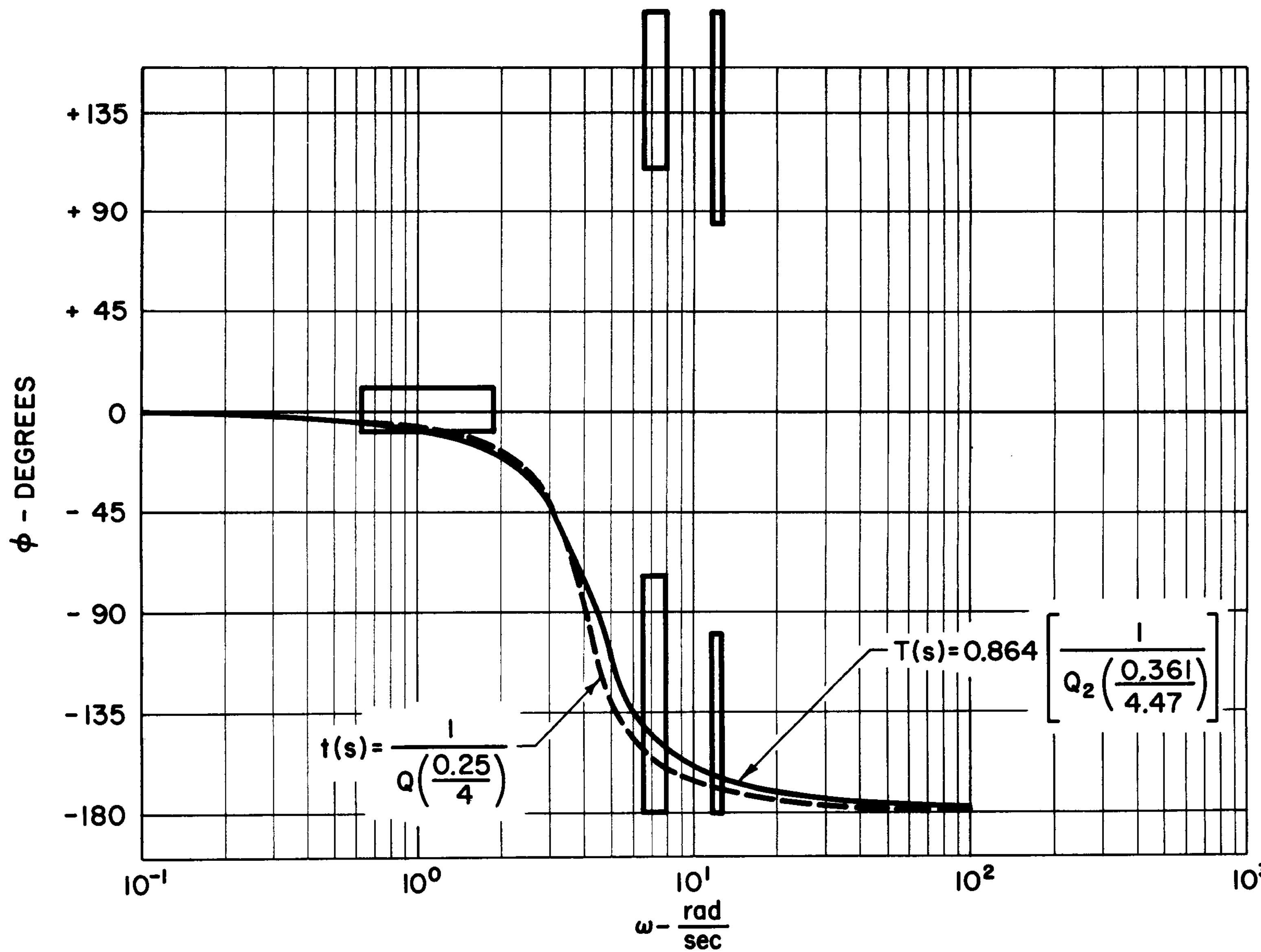


Figure VI-8. Phase Tolerance for Rate Gyro Filter

TABLE VI-3

SUMMARY OF SECOND SYNTHESIS ITERATIONS

Iteration	$X_1 = R_1$	$X_2 = L_1$	$X_3 = C_2$	$X_4 = R_3$	$X_5 = R_4$	Error
0	.6	.4	.15	.2	1.00	11.8259934968
1a	.4201705096	-.1943404530	-.0669695927	.1195369674	1.00	5.0217512361
$1b_1$	.4201705096	.4	.15	.1195369674	1.00	2.7676834456
2	.3413472702	.1743432705	.1500265175	.0870383568	1.00	1.7268784911
3	.2749490686	.2872532324	.2826097365	.0603316392	1.00	1.3135708257
4	.2606932953	.2182821915	.2444296171	.0603316392	1.00	.7201049826
5	.1395591754	.1772747031	.3661966219	.0338423400	1.00	.2313931353
$6a_2$	.1270746330	.1397549516	.3632264426	.0301622634	1.00	.2309822198
$6b_3$	101.6597064 Ω	111.80396128H	454.033053mfd.	24.12981072 Ω	8.00Ω	-----

1) Applied inequality constraint

2) Final normalized values

3) Final denormalized values

the classical synthesis procedure could be applied. The application of a digital computer to the classical technique is improbable because of the difficulty in writing a general program that would still only result in the calculation of ideal element values.

By formulating and minimizing the objective (error) function, greater emphasis and control is placed on satisfying the specifications with various network topologies at discrete frequencies of interest to the design engineer than by previous classical techniques. As indicated by Figures I-1 and VI-9, it is possible to mechanize the process presented in this report for digital computer applications, thereby simplifying the over-all synthesis procedure.

Minimum element count, topological configurations of interest, and probable computer mechanization further recommend this technique for synthesizing networks by minimizing real-valued functionals rather than using the more classical procedures.

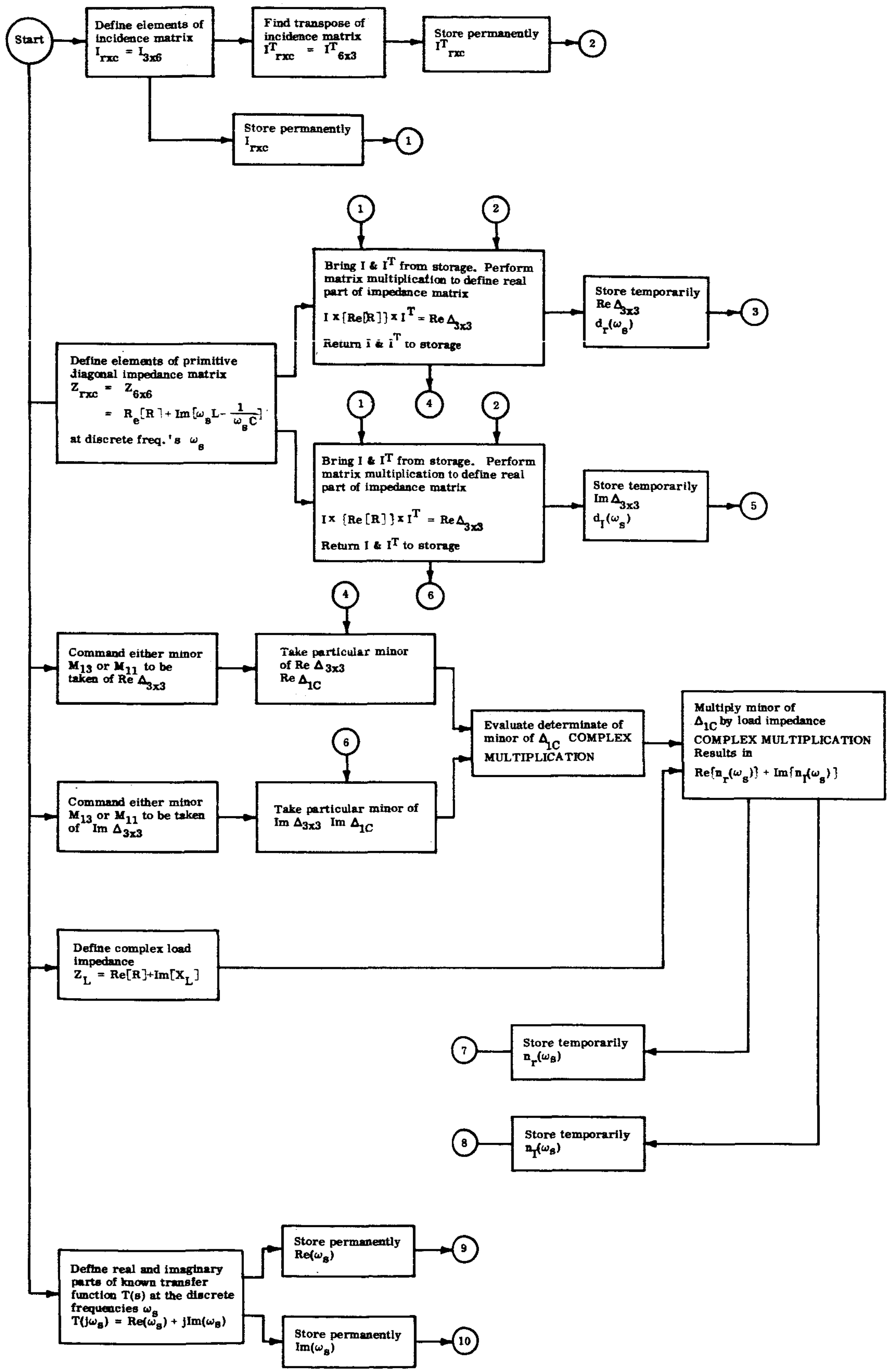


Figure VI-9. Flow Chart Outline of Main Program (Continued)

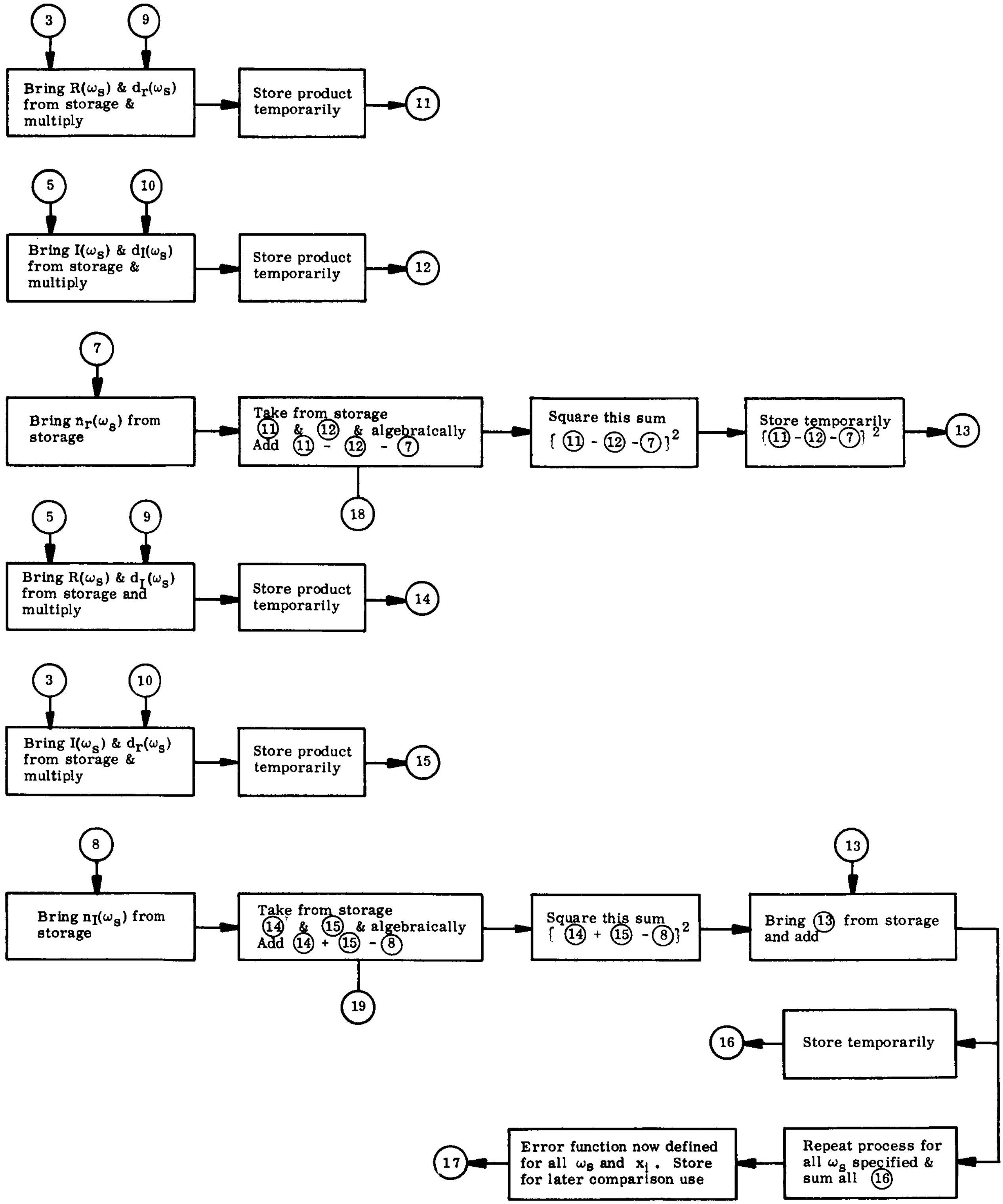


Figure VI-9 (Continued). Flow Chart Outline of Main Program

SECTION VII

SUMMARY AND CONCLUSIONS

A technique for synthesizing networks with resistive loads, resulting in normalized element values that have voltage transfer functions meeting the gain-phase compensation requirements set by NASA, has been presented. By minimization of the least squared error function using an iterative process such as the "tangent descent to a minimum" for a desk calculator, or the "gradient technique" for a digital computer, the normalized element values can be determined.

The concept of synthesizing networks from topological considerations was developed in Sections IV and V. A quadratic lag function synthesis was considered in detail in Section VI; a comparison was made between the classical synthesis technique and the one considered in this report. By considering topological configurations of interest, the mathematical process (which can be mechanized for digital computer application) yields less complex networks with non-ideal passive elements than those determined by classical synthesis techniques.

From the studies that have been completed, it would seem that further refinements are possible, starting with the over-all design procedure. The ultimate procedure is one that is completely computer-mechanized from the input gain-phase NASA requirement to the output network topology and realistic component values. Consequently, emphasis should be placed next on those subroutines that now offer a high degree of success, consistent with the presented theory of eventually being incorporated into the general approach.

A digital computer program should be written for 1) defining the objective function to include various network topologies of interest, and 2) mechanizing the iterative process used to determine the component values. Then, using the computer program as a tool, various areas of interest, such as convergence to the absolute minimum, initial approximations, resulting circuit configurations, and component values should be investigated.

The topology considerations should be expanded so that the best network configuration can easily be determined. A general topology should be defined that would be useful in generating a number of network configurations, each of which satisfies the requirements. Then the best network should be defined, using some criterion based on one or all of the following: ease of implementation, component values, reliability, number of components, or even individual preference.

Consistent with the input requirements, it should be possible to use this information to generate those specific topologies from the general considerations that would be the most advantageous to use in the iterative process of component value determinations.

The additional work in the above areas could hopefully lead to simpler, more straightforward computer procedures for the realization of any network that is generally optimum in terms of its configuration and passive elements.

SECTION VIII

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APPENDIX "A"

THE GRADIENT METHOD OF STEEPEST DESCENT TO A MINIMUM

Derivation of the expressions used in the application of the method of steepest descent for solving non-linear simultaneous equations will be derived.

Assume a choice of the x_i variables as an initial solution is made at which Φ of Equation (III-23) does not have a stationary value. The iteration method described will result in an improved set of values of the parameters. If so desired, they may be further improved (if the true minimum has not already been reached) by a continuation of the process, considering this improved set as a new initial solution.

If we regard x_i as rectangular Cartesian coordinates in a Euclidean n-space, for the trivial case n = 2, the geometrical description may be displayed accurately in a diagram. For n = 3, simple geometrical representation is not possible; but since geometry in n-space is analogous to geometry in 2-space or 3-space, the geometrical illustration in 2-space serves as a general guide to procedure and generalization to more variables will easily follow.

The method can be illustrated geometrically for the case of two variables either by means of a three-dimensional diagram in which two dimensions are used to accommodate the variables x and y and the third to accommodate the response surface Φ, or by two-dimensional diagrams in which the response surface is represented by contour lines of constant Φ.

In Figure A-1, let C_1 and C_3 represent curves of constant Φ, C_1 being the curve of intersection of the surface $\Phi(x, y)$ and the plane Φ_o. The curve C_2 is the intersection of the surface $\Phi(x, y)$ and the normal plane to C_1 at P_o; the point (x_o, y_o, P_o) is our initial approximation to the absolute minimum of

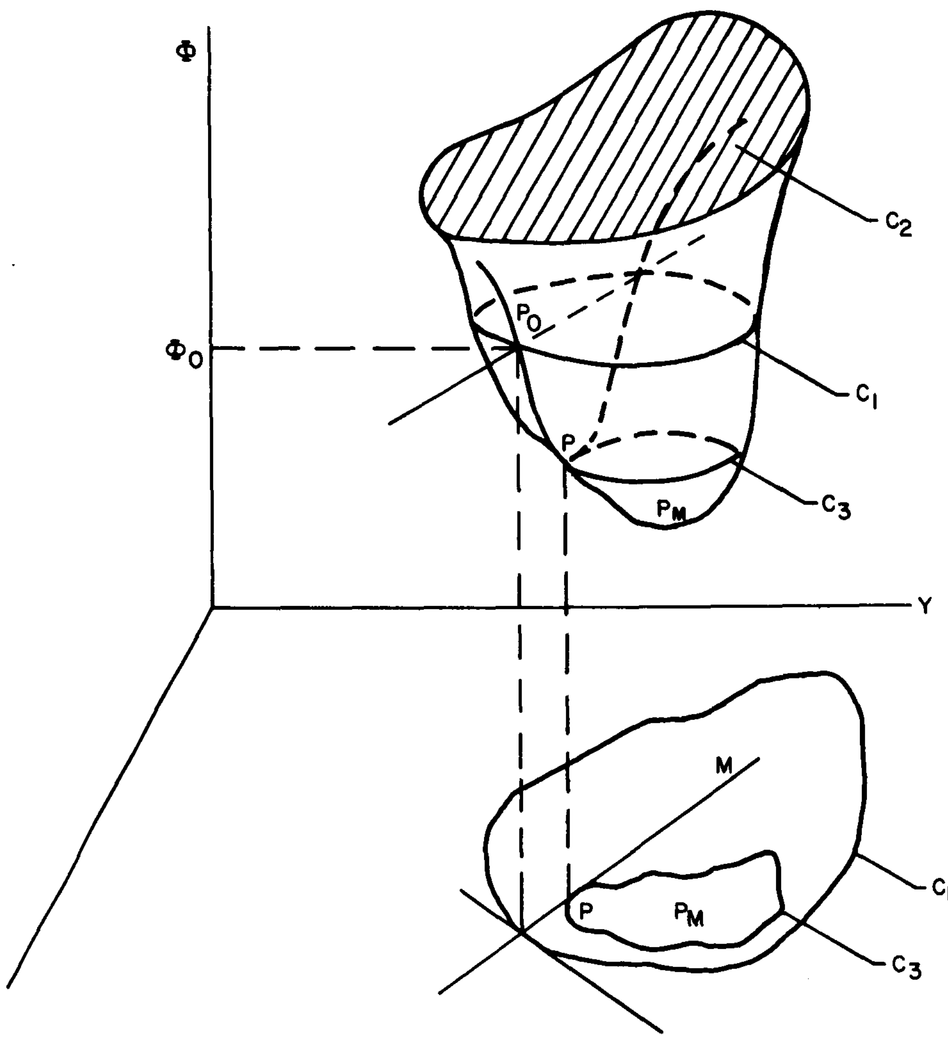

Figure A-1. Plot of the Surface $\Phi = \Phi(x, y)$

Φ, which in the depicted surface is the point P_M. We approach this point P_M by a stepwise process of the gradient technique, that is, the direction of steepest descent from our initial approximation is in the direction of the negative gradient of Φ which is in the direction of the curve C_2 extending to the local minimum at P. The negative gradient direction is normal to a contour illustrated as the projected axis in the x, y plane labeled n_1. The local minimum P is shown as the point of tangency, in the gradient direction n, to the contour C_3. This minimum P of C_2 is taken as a new approximation to P_M and the process is repeated.

In the plane (Φ, n) just now introduced, the curve C_2 has an equation $\Phi = \Phi(n)$. (Figure A-2)

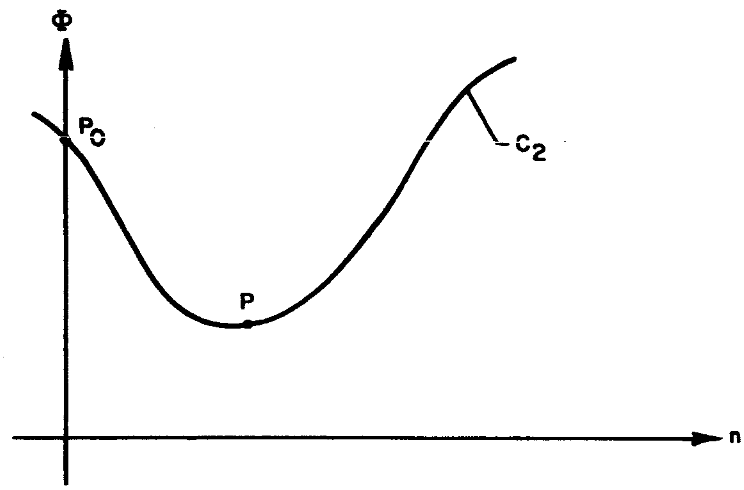

Figure A-2. Plot of C_2 in Coordinate Plane

A parabolic approximation to C_2 by Taylor's theorem gives:

$$\Phi = \Phi_o + \epsilon_n \Phi_n + \frac{\epsilon_n^2}{2} \Phi_{n,n} \qquad \text{(A-1)}$$

where:

a) Φ_n and $\Phi_{n,n}$ are the values of the indicated partial derivatives evaluated at the initial point P_o.

b) ϵ_n denotes the change in the x, y coordinates in the direction of the normal to C_1.

To determine the point P, we set the partial derivative of Equation (A-1), taken with respect to ϵ_n, to zero and solve the resulting expression as:

$$\epsilon_n = -\frac{\Phi_n}{\Phi_{n,n}} \qquad \text{(A-2)}$$

From an application of the theory of vector analysis, Φ_n in Equation (A-2) is found to be the directional derivative of Φ in a direction normal to the given curve C_1 through P_o and is often written in the form $\frac{\partial \Phi}{\partial n}$. Thus if $\bar{u}$ is a unit vector normal to C_1 (the unit vector which has the direction of grad Φ is $\frac{\text{grad } \Phi}{|\text{grad } \Phi|}$

we have:

$$\Phi_n = \frac{\partial \Phi}{\partial n} = \text{grad}\,\Phi \cdot \bar{u} = |\,\text{grad}\,\Phi\,| = \sqrt{\Phi_{x_o}^{\;2} + \Phi_{y_o}^{\;2}} \tag{A-3}$$

where:

$$\Phi_{x_o} = \frac{\partial \Phi_o}{\partial x}, \quad \Phi_{y_o} = \frac{\partial \Phi_o}{\partial y}$$

$\Phi_{n,n}$, the second partial of Φ_n is

$$\Phi_{n,n} = \frac{\partial \Phi_n}{\partial x}\frac{dx}{dn} + \frac{\partial \Phi_n}{\partial y}\frac{dy}{dn} \tag{A-4}$$

In Equation (A-4), $\frac{dx}{dn}$ and $\frac{dy}{dn}$ are required. Writing the equation of the n-axis using the point slope form of the equation of the straight line, namely,

$$y - y_o = \frac{dy}{dx}(x - x_o) \tag{A-5}$$

which is the equation of the tangent to C_1 at the point P_o, the line perpendicular to C_1 at P_o would be the normal (n-axis):

$$y - y_o = -\frac{dx}{dy}(x - x_o) \tag{A-6}$$

It follows, that for: $\Phi = \Phi_o$

$$\frac{dx}{dy} = -\frac{\Phi_{y_o}}{\Phi_{x_o}} = -\frac{\frac{\partial \Phi_o}{\partial y}}{\frac{\partial \Phi_o}{\partial x}} \tag{A-7}$$

Therefore, considering the point P of the curve C_3 projected to the n axis having coordinates x, y, the equation of the n axis in the (Φ, n) plane becomes:

$$y - y_o = \frac{\Phi_{y_o}}{\Phi_{x_o}}(x - x_o) \tag{A-8}$$

Equation (A-8) can be written in terms of n as:

$$n = \sqrt{\left(\frac{\Phi_{y_o}}{\Phi_{x_o}}\right)^2 (x - x_o)^2 + (x - x_o)^2} = \frac{(x-x_o)\sqrt{\Phi_{x_o}^{\;2} + \Phi_{y_o}^{\;2}}}{\Phi_{x_o}} \tag{A-9}$$

and then we get:

$$x = \frac{n\,\Phi_{x_o}}{\sqrt{\Phi_{x_o}^{\;2} + \Phi_{y_o}^{\;2}}} + x_o = n\,\frac{\Phi_{x_o}}{\Phi_n} + x_o \tag{A-10}$$

$$y = \frac{n\,\Phi_{y_o}}{\sqrt{\Phi_{x_o}^{\,2} + \Phi_{y_o}^{\,2}}} + y_o = n\,\frac{\Phi_{y_o}}{\Phi_n} + y_o \tag{A-11}$$

Therefore:

$$\frac{dx}{dn} = \frac{\Phi_{x_o}}{\Phi_n} \tag{A-12}$$

$$\frac{dy}{dn} = \frac{\Phi_{y_o}}{\Phi_n} \tag{A-13}$$

The remaining terms on the right side of Equation (A-4) are evaluated as:

$$\frac{\partial\,\Phi_n}{\partial\,x} = \frac{\Phi_{x_o}\,\Phi_{x_o x_o} + \Phi_{y_o}\,\Phi_{x_o y_o}}{\Phi_n} \tag{A-14}$$

and:

$$\frac{\partial\,\Phi_n}{\partial\,y} = \frac{\Phi_{x_o}\,\Phi_{x_o y_o} + \Phi_{y_o}\,\Phi_{y_o y_o}}{\Phi_n} \tag{A-15}$$

We thus evaluate Equation (A-4), substituting (A-12), (A-13), (A-14), (A-15), and get:

$$\Phi_{nn} = \frac{\left(\Phi_{x_o}\,\Phi_{x_o x_o} + \Phi_{y_o}\,\Phi_{x_o y_o}\right)}{\Phi_n}\,\frac{\Phi_{x_o}}{\Phi_n} + \frac{\left(\Phi_{x_o}\,\Phi_{x_o y_o} + \Phi_{y_o}\,\Phi_{y_o y_o}\right)}{\Phi_n}\,\frac{\Phi_{y_o}}{\Phi_n} \tag{A-16}$$

Equation (A-3) and Equation (A-16), substituted into Equation (A-2) gives:

$$\epsilon_n = -\frac{\Phi_n}{\Phi_{nn}} = -\frac{\Phi_n^{\,3}}{\left[\left(\Phi_{x_o}\Phi_{x_o x_o} + \Phi_{y_o}\Phi_{x_o y_o}\right)\Phi_{x_o} + \left(\Phi_{x_o}\Phi_{x_o y_o} + \Phi_{y_o}\Phi_{y_o y_o}\right)\Phi_{y_o}\right]} \tag{A-17}$$

Now Equation (A-17), written in terms of the increments in the coordinate directions is:

$$\epsilon_x = \epsilon_n \cos\,(n, x) \tag{A-18}$$

$$\epsilon_y = \epsilon_n \cos\,(n, y) \tag{A-19}$$

Where (n, z) and (n, y) are the angles between the n axis and the x and y axis, respectively, and:

$$\cos (n, x) = \frac{\Phi_{x_o}}{\Phi_n} \tag{A-20}$$

$$\cos (n, y) = \frac{\Phi_{y_o}}{\Phi_n} \tag{A-21}$$

Thus finally utilizing Equations (A-18), (A-20), and (A-17), we get:

$$\epsilon_x = - \frac{\Phi_{x_o}\left(\Phi_{y_o}^{\;2} + \Phi_{y_o}^{\;2}\right)}{\Phi_{x_o}\left(\Phi_{x_o x_o} + \Phi_{y_o}\Phi_{x_o y_o}\right) + \Phi_{y_o}\left(\Phi_{x_o}\Phi_{x_o y_o} + \Phi_{y_o}\Phi_{y_o y_o}\right)} \tag{A-22}$$

and for Equation (A-19), using Equations (A-21) and (A-17) gives:

$$\epsilon_y = - \frac{\Phi_{y_o}\left(\Phi_{x_o}^{\;2} + \Phi_{y_o}^{\;2}\right)}{\Phi_{y_o}\left(\Phi_{x_o}\Phi_{x_o x_o} + \Phi_{y_o}\Phi_{x_o y_o}\right) + \Phi_{y_o}\left(\Phi_{x_o}\Phi_{x_o y_o} + \Phi_{y_o}\Phi_{y_o y_o}\right)} \tag{A-23}$$

Equations (A-22) and (A-23) denote the changes in the x, y coordinate axis direction to the local minimum P.

Generalizing the above derivation to the case of x_i variables for $i = 1, 2 \ldots n$ we obtain:

$$\Phi_n = \frac{\partial \Phi}{\partial n} = | \text{ grad } \Phi \,| = \left\{ \sum_{i=1}^{N} (\Phi_i)^2 \right\}^{\frac{1}{2}} \tag{A-24}$$

where:

$$\Phi_i = \left[\frac{\partial \Phi}{\partial x_i}\right]_o$$

and:

$$\Phi_{nn} = \frac{\partial \Phi_n}{\partial n} \sum_{i=1}^{N} \frac{\partial \Phi_n}{\partial x_i} \frac{\partial x_i}{\partial n} = \frac{\left[\sum_{j,i=1}^{N} \Phi_{j_o} \Phi_{i_o} \Phi_{j_o i_o}\right]}{\Phi_n^{\;2}} \tag{A-25}$$

From Equations (A-24), (A-25), and (A-2) it follows that:

$$\epsilon_n = - \frac{\left\{ \sum_{i=1}^{N} (\Phi_i)^2 \right\}^{3/2}}{\left[\sum_{j,i=1}^{N} \Phi_{j_o} \Phi_{i_o} \Phi_{j_o i_o} \right]}$$

and consequently, the changes, ϵ_i in individual coordinates (x_i), for steepest descent are given by:

$$\epsilon_i = \epsilon_n \cos(n, x_i) = - \frac{\left(\Phi_{i_o} \sum_{i=1}^{N} \Phi_{i_o} \right)^2}{\left[\sum_{j,i=1}^{N} \Phi_{j_o} \Phi_{i_o} \Phi_{j_o i_o} \right]} \tag{A-26}$$

APPENDIX "B"

THE METHOD OF STEEPEST DESCENT ALONG A TANGENT

This method, developed by Booth, is a rapid method for improvement of the initial approximation, thus speeding up the process of convergence to a minimum, since it does not require any calculation of second derviatives. This is desirable for the application being considered where the variables are numerous and the second derivatives complicated. It involves only the calculation of the function to be minimized and its first derivatives.

In this approach, a linear approximation is made to the curve C_2 of Figure (A-1) as:

$$\Phi = \Phi_o + \epsilon_n \Phi_{n_o} \tag{B-1}$$

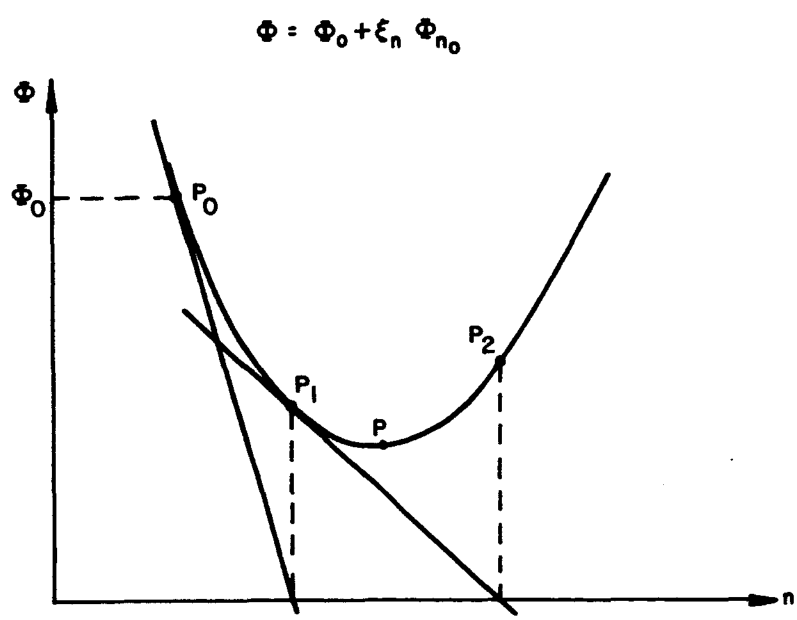

Figure B-1. Descent Along Tangent

In Equation (B-1), setting $\Phi = 0$, and solving, one obtains

$$\epsilon_n = - \frac{\Phi_o}{\Phi_{n_o}} \tag{B-2}$$

This is equivalent to moving down the tangent to the (Φ, n) curve at the point Φ_o, until the line intersects the n axis. The ordinate to the curve corresponding to the point of intersection of the tangent line and the n axis will be the approximation to $\Phi_{min.}$ on the next iteration. The procedure is repeated using the new approximation until Φ begins to increase, shown as point P_2 on Figure B-1, at which time it is necessary to revert to a more accurate technique such as Equation (A-1) or the Taylor series method.

The change in the individual coordinates is found utilizing the formulas of Equations B-2, A-18, A-19, A-20, A-21, and given as:

$$\epsilon_i = \epsilon_n \cos (n, x_i) = - \frac{\Phi_o \Phi_{i_o}}{\left[\sum_{i=1}^{n} (\Phi_{i_o})^2 \right]^{1/2}} \tag{B-3}$$

from which,

$$x_i^{p+1} = x_i^p + \epsilon_i^p \tag{B-4}$$

with x_i^p the variable in question, i^{th} dimension, p^{th} iteration.